Stirling number

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In mathematics, **Stirling numbers** arise in a variety of analytic and combinatorics problems. They are named after James Stirling, who introduced them in the 18th century. Two different sets of numbers bear this name: the Stirling numbers of the first kind and the Stirling numbers of the second kind.

Contents

Notation

Several different notations for Stirling numbers are in use. Stirling numbers of the first kind are written with a small *s*, and those of the second kind with a capital *S*. Stirling numbers of the second kind are never negative, but those of the first kind can be negative; hence, there are notations for the "unsigned Stirling numbers of the first kind", which are Stirling numbers without their signs. Common notations are:

$$s(n,k)$$

for ordinary (signed) Stirling numbers of the first kind,

$$c(n,k) = \left[{n \atop k}\right] = |s(n,k)|$$

for unsigned Stirling numbers of the first kind, and

$$S(n,k) = \left\{{n \atop k}\right\} = S_n^{(k)}$$

for Stirling numbers of the second kind.

Abramowitz and Stegun use an uppercase S and a blackletter S, respectively, for the first and second kinds of Stirling number. The notation of brackets and braces, in analogy to binomial coefficients, was introduced in 1935 by Jovan Karamata and promoted later by Donald Knuth. (The bracket notation conflicts with a common notation for Gaussian coefficients.[1]) The mathematical motivation for this type of notation, as well as additional Stirling number formulae, may be found on the page for Stirling numbers and exponential generating functions.

Stirling numbers of the first kind

The **Stirling numbers of the first kind** are the coefficients in the expansion

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k.$$

where $(x)_n$ (a Pochhammer symbol) denotes the falling factorial,

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1).$$

Note that $(x)_0 = 1$ because it is an empty product. Combinatorialists also sometimes use the notation $x^{\underline{n}}$ for the falling factorial, and $x^{\overline{n}}$ for the rising factorial.[2]

(Confusingly, the Pochhammer symbol that many use for *falling* factorials is used in special functions for *rising* factorials.)

The unsigned Stirling numbers of the first kind,

$$c(n,k) = \left[{n \atop k}\right] = |s(n,k)| = (-1)^{n-k}s(n,k)$$

(with a lower-case "*s*"), count the number of permutations of *n* elements with *k* disjoint cycles.

A few of the Stirling numbers of the first kind are illustrated by the table below, starting with row 0, column 0:

$$\begin{array}{rrrrrrr}
1 \\
0 & 1 \\
0 & -1 & 1 \\
0 & 2 & -3 & 1 \\
0 & -6 & 11 & -6 & 1 \\
0 & 24 & -50 & 35 & -10 & 1 \\
0 & -120 & 274 & -225 & 85 & -15 & 1
\end{array}$$

where

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k)$$

Stirling numbers of the second kind

Stirling numbers of the second kind count the number of ways to partition a set of *n* elements into *k* nonempty subsets. They are denoted by $S(n,k)$ or $\left\{{n \atop k}\right\}$.[3] The sum

$$\sum_{k=0}^{n} S(n,k) = B_n$$

is the *n*th Bell number.

Using falling factorials, we can characterize the Stirling numbers of the second kind by the identity

$$\sum_{k=0}^{n} S(n,k)(x)_k = x^n.$$

Lah numbers

The Lah numbers $L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}$ are sometimes called Stirling numbers of the third kind. For example, see Jozsef Sándor and Borislav Crstici, *Handbook of Number Theory II*, Volume 2 (https://books.google.com/books?id=B2WZkvmFKk8C&pg=PA464&lpg=PA464&dq=%22Stirling+numbers+of+the+third+kind%22&source=bl&ots=JhIJKIhaFH&sig=_0-CWfixhUoAuhh7DAo4fJco6y4&hl=en&ei=BKh2TfnBJ_KH0QGn17XZBg&sa=X&oi=book_result&ct=result&resnum=2&ved=0CCAQ6AEwAQ#v=onepage&q=%22Stirling%20numbers%20of%20the%20third%20kind%22&f=false).

Inversion relationships

The Stirling numbers of the first and second kinds can be considered inverses of one another:

$$\sum_{j=0}^{n} s(n,j)S(j,k) = \delta_{nk}$$

and

$$\sum_{j=0}^{n} S(n,j)s(j,k) = \delta_{nk},$$

where δ_{nk} is the Kronecker delta. These two relationships may be understood to be matrix inverse relationships. That is, let *s* be the lower triangular matrix of Stirling numbers of the first kind, whose matrix elements $s_{nk} = s(n,k).$ The inverse of this matrix is *S*, the lower triangular matrix of Stirling numbers of the second kind, whose entries are $S_{nk} = S(n,k).$ Symbolically, this is written

$$s^{-1} = S$$

Although *s* and *S* are infinite, so calculating a product entry involves an infinite sum, the matrix multiplications work because these matrices are lower triangular, so only a finite number of terms in the sum are nonzero.

A generalization of the inversion relationship gives the link with the Lah numbers $L(n,k)$:

$$(-1)^n L(n,k) = \sum_z (-1)^z s(n,z)S(z,k),$$

with the conventions $L(0,0) = 1$ and $L(n,k) = 0$ if $k > n$.

Symmetric formulae

Abramowitz and Stegun give the following symmetric formulae that relate the Stirling numbers of the first and second kind.[4]

$$s(n,k) = \sum_{j=0}^{n-k} (-1)^j \binom{n-1+j}{n-k+j}\binom{2n-k}{n-k-j} S(n-k+j,j)$$

and

$$S(n,k) = \sum_{j=0}^{n-k} (-1)^j \binom{n-1+j}{n-k+j}\binom{2n-k}{n-k-j} s(n-k+j,j).$$

See also

- Bell polynomials
- Cycles and fixed points
- Lah number
- Pochhammer symbol
- Polynomial sequence
- Stirling transform
- Touchard polynomials

References

1. Donald Knuth
2. Aigner, Martin (2007). "Section 1.2 - Subsets and Binomial Coefficients". *A Course In Enumeration*. Springer. p. 561. ISBN 3-540-39032-4.
3. Ronald L. Graham, Donald E. Knuth, Oren Patashnik (1988) *Concrete Mathematics*, Addison-Wesley, Reading MA. ISBN 0-201-14236-8, p. 244.
4. Goldberg, K.; Newman, M; Haynsworth, E. (1972), "Stirling Numbers of the First Kind, Stirling Numbers of the Second Kind", in Abramowitz, Milton; Stegun, Irene A., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 10th printing*, New York: Dover, pp. 824–825

- Milton Abramowitz and Irene A. Stegun, eds., *Handbook of Mathematical Functions (with Formulas, Graphs and Mathematical Tables)* (http://www.convertit.com/Go/ConvertIt/Reference/AMS55.ASP), U.S. Dept. of Commerce, National Bureau of Standards, Applied Math. Series 55, 1964, 1046 pages (10th Printing: December 1972) - Combinatorial Analysis, Table 24.4, Stirling Numbers of the Second Kind (author: Francis L. Miksa), p. 835.
- Victor Adamchik, "On Stirling Numbers and Euler Sums (http://www-2.cs.cmu.edu/~adamchik/articles/stirling.pdf)", Journal of Computational and Applied Mathematics **79** (1997) pp. 119–130.
- Arthur T. Benjamin, Gregory O. Preston, Jennifer J. Quinn, *A Stirling Encounter with Harmonic Numbers (http://www.math.hmc.edu/~benjamin/papers/harmonic.pdf)*, (2002) Mathematics Magazine, **75** (2) pp 95–103.
- Khristo N. Boyadzhiev, *Close encounters with the Stirling numbers of the second kind'* (http://www.maa.org/sites/default/files/pdf/upload_library/2/Boyadzhiev-2013.pdf)' (2012) Mathematics Magazine, **85** (4) pp 252–266.
- Louis Comtet, *Valeur de* s(n, k*)* (http://www.techniques-ingenieur.fr/page/af202niv10002/permutations.html#2.2), Analyse combinatoire, Tome second (page 51), Presses universitaires de France, 1970.
- Louis Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Reidel Publishing Company, Dordrecht-Holland/Boston-U.S.A., 1974.
- Hsien-Kuei Hwang (1995). "Asymptotic Expansions for the Stirling Numbers of the First Kind". *Journal of Combinatorial Theory, Series A*. **71** (2): 343–351. doi:10.1016/0097-3165(95)90010-1.
- D.E. Knuth, *Two notes on notation* (http://www-cs-faculty.stanford.edu/~knuth/papers/tnn.tex.gz) (TeX source).
- Francis L. Miksa (1901–1975), *Stirling numbers of the first kind* (http://www.jstor.org/stable/2002617), "27 leaves reproduced from typewritten manuscript on deposit in the UMT File", Mathematical Tables and Other Aids to Computation, vol. 10, no. 53, January 1956, pp. 37–38 (Reviews and Descriptions of Tables and Books, 7[I]).
- Dragoslav S. Mitrinović, *Sur les nombres de Stirling de première espèce et les polynômes de Stirling* (http://pefmath2.etf.bg.ac.rs/files/23/23.pdf), AMS 11B73_05A19, Publications de la Faculté d'Electrotechnique de l'Université de Belgrade, Série Mathématiques et Physique (ISSN 0522-8441), no. 23, 1959 (5.V.1959), pp. 1–20.
- John J. O'Connor and Edmund F. Robertson, *James Stirling (1692–1770)* (http://www-history.mcs.st-andrews.ac.uk/history/Biographies/Stirling.html), (September 1998).
- Sixdeniers, J. M.; Penson, K. A.; Solomon, A. I. (2001). "Extended Bell and Stirling Numbers From Hypergeometric Exponentiation" (PDF). *Journal of Integer Sequences*. **4**: 01.1.4..
- Spivey, Michael Z. (2007). "Combinatorial sums and finite differences". *Discr. Math.* **307** (24). pp. 3130–3146. doi:10.1016/j.disc.2007.03.052.
- "Sloane's A008275 : Stirling numbers of first kind". *The On-Line Encyclopedia of Integer Sequences*. OEIS Foundation.
- "Sloane's A008277 : Stirling numbers of 2nd kind". *The On-Line Encyclopedia of Integer Sequences*. OEIS Foundation.
- "Stirling numbers of the first kind, s(n,k)". *PlanetMath*..
- "Stirling numbers of the second kind, S(n,k)". *PlanetMath*..