Bell number

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In combinatorial mathematics, the **Bell numbers** count the number of partitions of a set. These numbers have been studied by mathematicians since the 19th century, and their roots go back to medieval Japan, but they are named after Eric Temple Bell, who wrote about them in the 1930s.

Starting with $B_0 = B_1 = 1$, the first few Bell numbers are:

> 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, 190899322, 1382958545, 10480142147, 82864869804, 682076806159, 5832742205057, ... (sequence A000110 in the OEIS).

The nth of these numbers, B_n, counts the number of different ways to partition a set that has exactly n elements, or equivalently, the number of equivalence relations on it. Outside of mathematics, the same number also counts the number of different rhyme schemes for n-line poems.[1]

As well as appearing in counting problems, these numbers have a different interpretation, as moments of probability distributions. In particular, B_n is the nth moment of a Poisson distribution with mean 1.

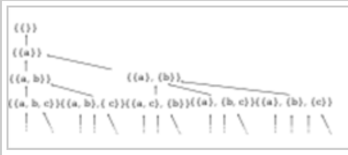
Contents

What these numbers count

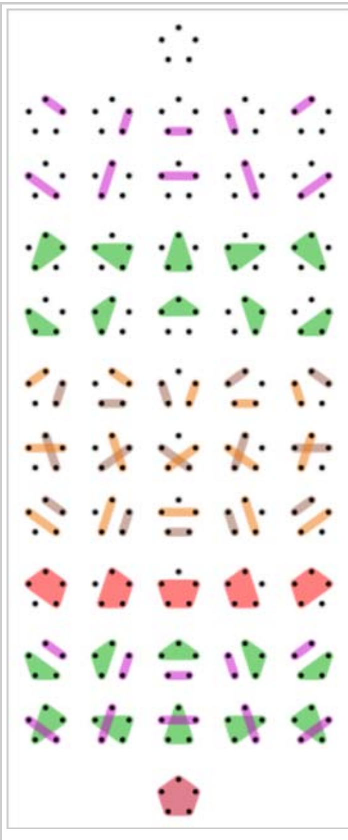
Set partitions

In general, B_n is the number of partitions of a set of size n. A partition of a set S is defined as a set of nonempty, pairwise disjoint subsets of S whose union is S. For example, $B_3 = 5$ because the 3-element set $\{a, b, c\}$ can be partitioned in 5 distinct ways:

$\{ \{a\}, \{b\}, \{c\} \}$
$\{ \{a\}, \{b, c\} \}$
$\{ \{b\}, \{a, c\} \}$
$\{ \{c\}, \{a, b\} \}$
$\{ \{a, b, c\} \}$.



Partitions of sets can be arranged in a partial order, showing that each partition of a set of size n "uses" one of the partitions of a set of size n-1.



The 52 partitions of a set with 5 elements

B_0 is 1 because there is exactly one partition of the empty set. Every member of the empty set is a nonempty set (that is vacuously true), and their union is the empty set. Therefore, the empty set is the only partition of itself. As suggested by the set notation above, we consider neither the order of the partitions nor the order of elements within each partition. This means that the following partitionings are all considered identical:

$\{ \{b\}, \{a, c\} \}$
$\{ \{a, c\}, \{b\} \}$
$\{ \{b\}, \{c, a\} \}$
$\{ \{c, a\}, \{b\} \}$.

If, instead, different orderings of the sets are considered to be different partitions, then the number of these ordered partitions is given by the ordered Bell numbers.

Factorizations

If a number N is a squarefree number (meaning that it is the product of some number n of distinct prime numbers), then B_n gives the number of different multiplicative partitions of N. These are factorizations of N into numbers greater than one, treating two factorizations as the same if they have the same factors in a different order.[2] For instance, 30 is the product of the three primes 2, 3, and 5, and has five factorizations:

$$30 \times 1 = 2 \times 15 = 3 \times 10 = 5 \times 6 = 2 \times 3 \times 5$$

Rhyme schemes

The Bell numbers also count the rhyme schemes of an n-line poem or stanza. A rhyme scheme describes which lines rhyme with each other, and so may be interpreted as a partition of the set of lines into rhyming subsets. Rhyme schemes are usually written as sequence of Roman letters, one per line, with rhyming lines given the same letter as each other, and with the first lines in each rhyming set labeled in alphabetical order. Thus, the 15 possible four-line rhyme schemes are AAAA, AAAB, AABA, AABB, AABC, ABAA, ABAB, ABAC, ABBA, ABBB, ABBC, ABCA, ABCB, ABCC, and ABCD.[1]

Permutations

The Bell numbers come up in a card shuffling problem mentioned in the addendum to Gardner (1978). If a deck of n cards is shuffled by repeatedly removing the top card and reinserting it anywhere in the deck (including its original position at the top of the deck), with exactly n repetitions of this operation, then there are n^n different shuffles that can be performed. Of these, the number that return the deck to its original sorted order is exactly B_n. Thus, the probability that the deck is in its original order after shuffling it in this way is B_n/n^n, which is significantly larger than the $1/n!$ probability that would describe a uniformly random permutation of the deck.

Related to card shuffling are several other problems of counting special kinds of permutations that are also answered by the Bell numbers. For instance, the nth Bell number equals number of permutations on n items in which no three values that are in sorted order have the last two of these three consecutive. In a notation for generalized permutation patterns where values that must be consecutive are written adjacent to each other, and values that can appear non-consecutively are separated by a dash, these permutations can be described as the permutations that avoid the pattern 1-23. The permutations that avoid the generalized patterns 12-3, 32-1, 3-21, 1-32, 3-12, 21-3, and 23-1 are also counted by the Bell numbers.[3] The permutations in which every 321 pattern (without restriction on consecutive values) can be extended to a 3241 pattern are also counted by the Bell numbers.[4] However, the Bell numbers grow too quickly to count the permutations that avoid a pattern that has not been generalized in this way: by the (now proven) Stanley–Wilf conjecture, the number of such permutations is singly exponential, and the Bell numbers have a higher asymptotic growth rate than that.

Triangle scheme for calculations

The triangular array whose right-hand diagonal sequence consists of Bell numbers

The Bell numbers can easily be calculated by creating the so-called Bell triangle, also called **Aitken's array** or the **Peirce triangle** after Alexander Aitken and Charles Sanders Peirce.[5]

1. Start with the number one. Put this on a row by itself. ($x_{0,1} = 1$)
2. Start a new row with the rightmost element from the previous row as the leftmost number ($x_{i,1} \leftarrow x_{i-1,r}$ where r is the last element of (i-1)-th row)
3. Determine the numbers not on the left column by taking the sum of the number to the left and the number above the number to the left, that is, the number diagonally up and left of the number we are calculating $(x_{i,j} \leftarrow x_{i,j-1} + x_{i-1,j-1})$
4. Repeat step three until there is a new row with one more number than the previous row (Do Step 3 until $j = r + 1$)
5. The number on the left hand side of a given row is the *Bell number* for that row. ($B_i \leftarrow x_{i,1}$)

Here are the first five rows of the triangle constructed by these rules:

```
 1
 1   2
 2   3   5
 5   7  10  15
15  20  27  37  52
```

The Bell numbers appear on both the left and right sides of the triangle.

Properties

Summation formulas

The Bell numbers satisfy a recurrence relation involving binomial coefficients:[6]

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

It can be explained by observing that, from an arbitrary partition of $n + 1$ items, removing the set containing the first item leaves a partition of a smaller set of k items for some number k that may range from 0 to n. There are $\binom{n}{k}$ choices for the k items that remain after one set is removed, and B_k choices of how to partition them.

A different summation formula represents each Bell number as a sum of Stirling numbers of the second kind

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}.$$

The Stirling number $\left\{ {n \atop k} \right\}$ is the number of ways to partition a set of cardinality n into exactly k nonempty subsets. Thus, in the equation relating the Bell numbers to the Stirling numbers, each partition counted on the left hand side of the equation is counted in exactly one of the terms of the sum on the right hand side, the one for which k is the number of sets in the partition.[7]

Spivey (2008) has given a formula that combines both of these summations:

$$B_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} \left\{ {m \atop j} \right\} \binom{n}{k} j^{n-k} B_k.$$

Generating function

The exponential generating function of the Bell numbers is

$$B(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = e^{e^x - 1}.$$

In this formula, the summation in the middle is the general form used to define the exponential generating function for any sequence of numbers, and the formula on the right is the result of performing the summation in the specific case of the Bell numbers.

One way to derive this result uses analytic combinatorics, a style of mathematical reasoning in which sets of mathematical objects are described by formulas explaining their construction from simpler objects, and then those formulas are manipulated to derive the combinatorial properties of the objects. In the language of analytic combinatorics, a set partition may be described as a set of nonempty urns into which elements labelled from 1 to n have been distributed, and the combinatorial class of all partitions (for all n) may be expressed by the notation

$$\mathrm{S{\scriptsize ET}}(\mathrm{S{\scriptsize ET}}_{\geq 1}(\mathcal{Z})).$$

Here, $\mathcal{Z}$ is a combinatorial class with only a single member of size one, an element that can be placed into an urn. The inner $\mathrm{S{\scriptsize ET}}_{\geq 1}$ operator describes a set or urn that contains one or more labelled elements, and the outer $\mathrm{S{\scriptsize ET}}$ describes the overall partition as a set of these urns. The exponential generating function may then be read off from this notation by translating the $\mathrm{S{\scriptsize ET}}$ operator into the exponential function and the nonemptiness constraint ≥1 into subtraction by one.[8]

An alternative method for deriving the same generating function uses the recurrence relation for the Bell numbers in terms of binomial coefficients to show that the exponential generating function satisfies the differential equation $B'(x) = e^x B(x)$. The function itself can be found by solving this equation.[9]

Moments of probability distributions

The Bell numbers satisfy Dobinski's formula[10]

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

This formula can be derived by expanding the exponential generating function using the Taylor series for the exponential function, and then collecting terms with the same exponent.[8] It allows B_n to be interpreted as the nth moment of a Poisson distribution with expected value 1.

The nth Bell number is also the sum of the coefficients in the nth complete Bell polynomial, which expresses the nth moment of any probability distribution as a function of the first n cumulants.

Modular arithmetic

The Bell numbers obey Touchard's congruence: If p is any prime number then[11]

$$B_{p+n} \equiv B_n + B_{n+1} \pmod{p}$$

or, generalizing[12]

$$B_{p^m+n} \equiv mB_n + B_{n+1} \pmod{p}.$$

Because of Touchard's congruence, the Bell numbers are periodic modulo p, for every prime number p; for instance, for $p = 2$, the Bell numbers repeat the pattern odd-odd-even with period three. The period of this repetition, for an arbitrary prime number p, must be a divisor of

$$\frac{p^p - 1}{p - 1}$$

and for all prime $p \leq 101$ and $p = 113, 163, 167$, or 173 it is exactly this number (sequence A001039 in the OEIS).[13]

The period of the Bell numbers to modulo n are

> 1, 3, 13, 12, 781, 39, 137257, 24, 39, 2343, 28531167061, 156, 25239592216021, 411771, 10153, 48, 51702516367896047761, 39, 109912203092239643840221, 9372, 1784341, 85593501183, 949112181811268728834319677753, 312, 3905, 75718776648063, 117, 1647084, 91703076898614683377208150526107718802981, 30459, 568972471024107865287021434301977158534824481, 96, 370905171793, 155107549103688143283, 107197717, 156, ... (sequence A054767 in the OEIS)

Integral representation

An application of Cauchy's integral formula to the exponential generating function yields the complex integral representation

$$B_n = \frac{n!}{2\pi i e} \int_{\gamma} \frac{e^{e^z}}{z^{n+1}} \, dz.$$

Some asymptotic representations can then be derived by a standard application of the method of steepest descent.[14]

Log-concavity

The Bell numbers form a logarithmically convex sequence. Dividing them by the factorials, $B_n/n!$, gives a logarithmically concave sequence.[15]

Growth rate

Several asymptotic formulas for the Bell numbers are known. In Berend & Tassa (2010) the following bounds were established:

$$B_n < \left(\frac{0.792n}{\ln(n+1)} \right)^n \text{ for all positive integers } n;$$

moreover, if $\varepsilon > 0$ then for all $n > n_0(\varepsilon)$,

$$B_n < \left(\frac{e^{-0.6+\varepsilon} n}{\ln(n+1)} \right)^n$$

where $n_0(\varepsilon) = \max\left\{e^4, d^{-1}(\varepsilon)\right\}$ and $d(x) := \ln\ln(x+1) - \ln\ln x + \frac{1+e^{-1}}{\ln x}$. The Bell numbers can also be approximated using the Lambert W function, a function with the same growth rate as the logarithm, as [16]

$$B_n \sim \frac{1}{\sqrt{n}} \left(\frac{n}{W(n)} \right)^{n+\frac{1}{2}} \exp\left(\frac{n}{W(n)} - n - 1 \right).$$

Moser & Wyman (1955) established the expansion

$$B_{n+h} = \frac{(n+h)!}{W(n)^{n+h}} \times \frac{\exp(e^{W(n)} - 1)}{(2\pi B)^{1/2}} \times \left(1 + \frac{P_0 + hP_1 + h^2 P_2}{e^{W(n)}} + \frac{Q_0 + hQ_1 + h^2 Q_2 + h^3 Q_3 + h^4 Q_4}{e^{2W(n)}} + O(e^{-3W(n)}) \right)$$

uniformly for $h = O(\ln(n))$ as $n \to \infty$, where B and each P_i and Q_i are known expressions in $W(n)$.[17]

The asymptotic expression

$$\frac{\ln B_n}{n} = \ln n - \ln\ln n - 1 + \frac{\ln\ln n}{\ln n} + \frac{1}{\ln n} + \frac{1}{2}\left(\frac{\ln\ln n}{\ln n} \right)^2 + O\left(\frac{\ln\ln n}{(\ln n)^2} \right)$$
$$\text{as } n \to \infty$$

was established by de Bruijn (1981).

Bell primes

Gardner (1978) raised the question of whether infinitely many Bell numbers are also prime numbers. The first few Bell numbers that are prime are:

> 2, 5, 877, 27644437, 35742549198872617291353508656626642567, 359334085968622831041960188598043661065388726959079837 (sequence A051131 in the OEIS)

corresponding to the indices 2, 3, 7, 13, 42 and 55 (sequence A051130 in the OEIS).

The next **Bell prime** is B_{2841}, which is approximately $9.30740105 \times 10^{6538}$.[18] As of 2006, it is the largest known prime Bell number. Phil Carmody showed it was a probable prime in 2002. After 17 months of computation with Marcel Martin's ECPP program Primo, Ignacio Larrosa Cañestro proved it to be prime in 2004. He ruled out any other possible primes below B_{6000}, later extended to B_{30447} by Eric Weisstein.[19]

History

The Bell numbers are named after Eric Temple Bell, who wrote about them in 1938, following up a 1934 paper in which he studied the Bell polynomials.[20] Bell did not claim to have discovered these numbers; in his 1938 paper, he wrote that the Bell numbers "have been frequently investigated" and "have been rediscovered many times". Bell cites several earlier publications on these numbers, beginning with Dobiński (1877) which gives Dobinski's formula for the Bell numbers. Bell called these numbers "exponential numbers"; the name "Bell numbers" and the notation B_n for these numbers was given to them by Becker & Riordan (1948).[21]

The first exhaustive enumeration of set partitions appears to have occurred in medieval Japan, where (inspired by the popularity of the book The Tale of Genji) a parlor game called *genji-ko* sprang up, in which guests were given five packets of incense to smell and were asked to guess which ones were the same as each other and which were different. The 52 possible solutions, counted by the Bell number B_5, were recorded by 52 different diagrams, which were printed above the chapter headings in some editions of The Tale of Genji.[22]

In Srinivasa Ramanujan's second notebook, he investigated both Bell polynomials and Bell numbers.[23] Early references for the Bell triangle, which has the Bell numbers on both of its sides, include Peirce (1880) and Aitken (1933).

See also

- Touchard polynomials

Notes

1. Gardner (1978).
2. Williams (1945) credits this observation to Silvio Minetola's *Principii di Analisi Combinatoria* (1909).
3. Claesson (2001).
4. Callan (2006).
5. "Sloane's A011971 : Aitken's array". *The On-Line Encyclopedia of Integer Sequences*. OEIS Foundation.
6. Wilf (1994), p. 23.
7. Conway & Guy (1996).
8. Flajolet & Sedgewick (2009).
9. Rota (1964); Wilf (1994), pp. 20–23; Bender & Williamson (2006).
10. Dobiński (1877); Rota (1964); Bender & Williamson (2006).
11. Becker & Riordan (1948).
12. Hurst & Schultz (2009).
13. Williams (1945); Wagstaff (1996).
14. Simon, Barry (2010). "Example 15.4.6 (Asymptotics of Bell Numbers)". *Complex Analysis* (PDF). pp. 772–774.
15. Engel (1994); Canfield (1995); Asai, Kubo & Kuo (2000).
16. Lovász (1993).
17. Canfield, Rod (July 1994). "The Moser-Wyman expansion of the Bell numbers" (PDF). Retrieved 2013-10-24.
18. "93074010508593618333...(6499 other digits)...83885253703080601131". *5000 Largest Known Primes, The Prime Database*. Retrieved 2013-10-24.
19. Weisstein, Eric W. "Integer Sequence Primes". *MathWorld*.
20. Bell (1934); Bell (1938).
21. Rota (1964). However, Rota gives an incorrect date, 1934, for Becker & Riordan (1948).
22. Knuth (2013). Gardner (1978) and Berndt (2011) also mention the connection between Bell numbers and The Tale of Genji, in less detail.
23. Berndt (2011).

References

- Asai, Nobuhiro; Kubo, Izumi; Kuo, Hui-Hsiung (2000). "Bell numbers, log-concavity, and log-convexity". *Acta Applicandae Mathematicae*. **63** (1-3): 79–87. arXiv:math/0104137. doi:10.1023/A:1010738827855. MR 1831247.
- Aitken, A. C. (1933). "A problem in combinations". *Mathematical Notes*. **28**: 18–23. doi:10.1017/S1757748900002334.
- Becker, H. W.; Riordan, John (1948). "The arithmetic of Bell and Stirling numbers". *American Journal of Mathematics*. **70**: 385–394. doi:10.2307/2372336. JSTOR 2372336..
- Bell, E. T. (1934). "Exponential polynomials". *Annals of Mathematics*. **35**: 258–277. doi:10.2307/1968431. JSTOR 1968431..
- Bell, E. T. (1938). "The iterated exponential integers". *Annals of Mathematics*. **39**: 539–557. doi:10.2307/1968633. JSTOR 1968633..
- Bender, Edward A.; Williamson, S. Gill (2006). "Example 11.7, Set Partitions". *Foundations of Combinatorics with Applications* (PDF). Dover. pp. 319–320. ISBN 0-486-44603-4.
- Berend, D.; Tassa, T. (2010). "Improved bounds on Bell numbers and on moments of sums of random variables". *Probability and Mathematical Statistics*. **30** (2): 185–205.
- Berndt, Bruce C. (2011). "Ramanujan Reaches His Hand From His Grave To Snatch Your Theorems From You" (PDF). *Asia Pacific Mathematics Newsletter*. **1** (2): 8–13.
- de Bruijn, N.G. (1981). *Asymptotic methods in analysis* (3rd ed.). Dover. p. 108.
- Callan, David (2006). "A combinatorial interpretation of the eigensequence for composition". *Journal of Integer Sequences*. **9** (1): 06.1.4. arXiv:math/0507169. Bibcode:2005math......7169C. MR 2193154.
- Canfield, E. Rodney (1995). "Engel's inequality for Bell numbers". *Journal of Combinatorial Theory*. Series A. **72** (1): 184–187. doi:10.1016/0097-3165(95)90033-0. MR 1354972.
- Claesson, Anders (2001). "Generalized pattern avoidance". *European Journal of Combinatorics*. **22** (7): 961–971. doi:10.1006/eujc.2001.0515. MR 1857258.
- Conway, John Horton; Guy, Richard K. (1996). "Famous Families of Numbers: Bell Numbers and Stirling Numbers". *The Book of Numbers*. Copernicus Series. Springer. pp. 91–94. ISBN 9780387979939.
- Dobiński, G. (1877). "Summirung der Reihe $\sum \frac{n^m}{n!}$ für m = 1, 2, 3, 4, 5, …". *Grunert's Archiv*. **61**: 333–336.
- Engel, Konrad (1994). "On the average rank of an element in a filter of the partition lattice". *Journal of Combinatorial Theory*. Series A. **65** (1): 67–78. doi:10.1016/0097-3165(94)90038-8. MR 1255264.
- Flajolet, Philippe; Sedgewick, Robert (2009). "II.3 Surjections, set partitions, and words". *Analytic Combinatorics* (PDF). Cambridge University Press. pp. 106–119.
- Gardner, Martin (1978). "The Bells: versatile numbers that can count partitions of a set, primes and even rhymes". *Scientific American*. **238**: 24–30. doi:10.1038/scientificamerican0578-24. Reprinted with an addendum as "The Tinkly Temple Bells", Chapter 2 of *Fractal Music, Hypercards, and more ... Mathematical Recreations from Scientific American*, W. H. Freeman, 1992, pp. 24–38
- Hazewinkel, Michiel, ed. (2001), "Bell numbers", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- Hurst, Greg; Schultz, Andrew (2009). "An elementary (number theory) proof of Touchard's congruence". arXiv:0906.0696 [math.CO].
- Knuth, Donald E. (2013). "Two thousand years of combinatorics". In Wilson, Robin; Watkins, John J. *Combinatorics: Ancient and Modern*. Oxford University Press. pp. 7–37.
- Lovász, L. (1993). "Section 1.14, Problem 9". *Combinatorial Problems and Exercises* (2nd ed.). Amsterdam, Netherlands: North-Holland. p. 17. Zbl 0785.05001.
- Moser, Leo; Wyman, Max (1955). "An asymptotic formula for the Bell numbers". *Transactions of the Royal Society of Canada, Section III*. **49**: 49–54. MR 0078489.
- Peirce, C. S. (1880). "On the algebra of logic". *American Journal of Mathematics*. **3** (1): 15–57. doi:10.2307/2369442. JSTOR 2369442..
- Rota, Gian-Carlo (1964), "The number of partitions of a set", *American Mathematical Monthly*, **71** (5): 498–504, doi:10.2307/2312585, MR 0161805
- Spivey, Michael Z. (2008). "A generalized recurrence for Bell numbers" (PDF). *Journal of Integer Sequences*. **11** (2): Article 08.2.5, 3. MR 2420912.
- Wagstaff, Samuel S. (1996). "Aurifeuillian factorizations and the period of the Bell numbers modulo a prime". *Mathematics of Computation*. **65** (213): 383–391. Bibcode:1996MaCom..65..383W. doi:10.1090/S0025-5718-96-00683-7. MR 1325876.
- Wilf, Herbert S. (1994). *Generatingfunctionology* (PDF) (2nd ed.). Boston, MA: Academic Press. ISBN 0-12-751956-4. Zbl 0831.05001.
- Williams, G. T. (1945). "Numbers generated by the function $e^{e^x - 1}$". *American Mathematical Monthly*. **52**: 323–327. JSTOR 2305292. MR 0012612.

External links

- Robert Dickau. "Diagrams of Bell numbers".
- Weisstein, Eric W. "Bell Number". *MathWorld*.
- Gottfried Helms. "Further properties & Generalization of Bell-Numbers" (PDF).