GRAPH MAXIMUM PROBLEM
(FROM THE CHINA TEAM SELECTION TEST 2018)

Let G be a simple graph with 100 vertices such that for each vertex u, there exists a vertex $v \in N(u)$ and $N(u) \cap N(v) = \emptyset$. Try to find the maximum possible number of edges in G. The $N(.)$ refers to the neighborhood.

Consider the graph H on the 100 vertices where we connect two vertices u and v if in G, u and v are connected and $N(u) \cap N(v) = \emptyset$. The condition tells us that there are no isolated vertices in H. First, we partition the vertices into groups $A_1, A_2, \ldots, A_k$ so that the H-induced subgraphs on A_i is a star graph for all i. This is easy to do by the following algorithm: If there are no vertices of degree 1, then remove any two connected vertices and put them in a group. Otherwise, take a vertex u of degree 1 and look at the sole vertex v it is connected to. Remove v and all vertices of degree 1 it is connected to and put them in a group.

Now suppose we have our groups $A_1, A_2, \ldots, A_k$, and let the vertex in A_i that does not have degree 1 be v_i (if $|A_i| = 2$, let either vertex be v_i). Let $|A_i| = x_i + 1$ for all i, so x_i is the number of vertices in degree 1 (different from v_i if $|A_i| = 2$) in A_i.

The key claim here is that there are at most $x_i x_j + 1$ edges between A_i and A_j for any i, j. Indeed, note that if $u \in A_i$, $u \neq v_i$, and u is connected to v_j, then u cannot be connected to any other vertex w in A_j. This is because otherwise, $u \in N(v_j) \cap N(w)$, a contradiction. Now, if u is connected to v_j, then replace the edge between u and v_j with edges between u and the rest of the vertices in A_j. This does not decrease the number of edges between A_i and A_j. At the end, v_i and v_j can only be adjacent to each other. The other edges form a subgraph of the K_{a_i, a_j} between $A_i \setminus \{v_i\}$ and $A_j \setminus \{v_j\}$, so the claim is proven.

Also, within A_i there are exactly x_i edges. Indeed, v_i is connected to the rest of the vertices in A_i and if $u_1, u_2 \in A_i$ are connected with $u_1, u_2 \neq v_i$, then $u_1 \in N(v_i) \cap N(u_2)$, a contradiction. Summing over all A_i and A_i, A_j gives that there are at most $\binom{k}{2} + \sum_{1 \le i < j \le k} x_i x_j + \sum_{1 \le i \le k} x_i$ edges. This is achievable by taking drawing an edge between v_i, v_j for all i, j, v_i and the rest of the vertices in A_i for all i, and a $K_{x_1, x_2, \ldots, x_k}$ on $A_1 \backslash \{v_1\}, A_2 \backslash \{v_2\}, \ldots, A_k \backslash \{v_k\}$. It is easy to check that this graph satisfies the conditions.

Hence, we want to maximize $\binom{k}{2} + 100 - k + \binom{100-k}{2} - \sum_{i=1}^{k} \binom{x_k}{2}$ over all positive integers $k, x_1, x_2, \ldots, x_k$ with $x_1 + x_2 + \cdots + x_k = 100 - k$.

Note that if we let the x_i instead be positive reals, then this function is at most

$$f(k) = \binom{k}{2} + 100 - k + \binom{100-k}{2} - k\binom{\frac{100}{k} - 1}{2}$$

by the convexity of $\binom{x}{2}$. It is not difficult to see that this is increasing on $k \le 7$ and decreasing on $9 \le k \le 50$, and furthermore we know that $f(7) = \frac{26745}{7} < 3822$ and $f(9) = \frac{34267}{9} < 3822$. So if $k \neq 8$, the number of edges is less than 3822. Now note that if $k = 8$, we would like to minimize the expression $\sum_{i=1}^{8} \binom{x_i}{2}$ subject to $x_1 + ... + x_8 = 92$. Again by convexity, this is minimized at $(x_1, ..., x_8) = (12, 12, 12, 12, 11, 11, 11, 11)$ (and permutations), so the maximum of the given expression for $k = 8$ is $\binom{8}{2} + 92 + \binom{92}{2} - 4\binom{12}{2} - 4\binom{11}{2} = 3822$, which is greater than the possible values for $k \neq 8$ and thus the answer is 3822.

The equality case is a $K_{12,12,12,12,11,11,11,11}$ (colored with 8 colors) along with a K_8 such that each element of the K_8 is connected to all the elements of a distinct color.