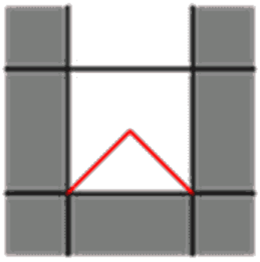
CENTER OF THE GRID IS AN ENDPOINT TO A PATH (FROM THE CHINA TEAM SELECTION TEST 2017)

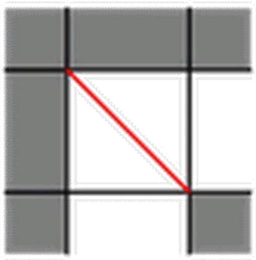
Every cell of a 2017×2017 grid is colored either black or white, such that every cell has at least one side in common with another cell of the same color. Let V_1 be the set of all black cells, V_2 be the set of all white cells. For set $V_i (i = 1, 2)$, if two cells share a common side, draw an edge with the centers of the two cells as endpoints, obtaining graphs G_i. If both G_1 and G_2 are connected paths (no cycles, no splits), prove that the center of the grid is one of the endpoints of G_1 or G_2.

Solution:

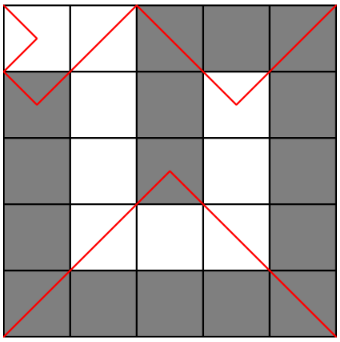
For each endpoint of the path, draw a red V shape pointing to the two corners not contained by the adjacent square of the path. (The touching squares drawn below don't have to exist; we can draw this V in a corner or edge square.)



For each turn in the path, draw a red diagonal line through the shared corner of the adjacent squares. (Again, not all touching squares have to exist.)



Or equivalently, at any vertex (edge and corner included) of the grid touching exactly one square of a given color, we draw a red edge toward the center of that square. If it's an interior vertex touching three squares of the other color, we also draw a red edge toward the center of the middle square of those three. Note it is not possible for an interior vertex to touch exactly two squares of the same color which are diagonally adjacent, since the paths would have to intersect. From this characterization, it is clear that the red edges form their own paths that do not split off or intersect and which can only start or end at a corner of the grid. Each corner of the grid also must start or end a red path.



We conclude there are exactly two disjoint red paths. Since they don't intersect, each connects two adjacent corners of the grid (i.e. corners which are the endpoints of a side, as opposed to a diagonal).

Now color the vertices (all 2018^2) blue and green like a checkerboard. Each red path switches which color of vertices it is travelling on only when it hits a V shape corresponding to an endpoint of a black or white path. Since the red paths connect oppositely colored corners, each red path contains an odd number of these V shapes. Therefore, there is a red path containing exactly one V shape. Because red paths can only turn at V shapes or by bouncing off the edge of the board, this red path's only V shape has to be in the center square where both corners will hit it, so we're done.