USA IMO TST 2019 Solutions

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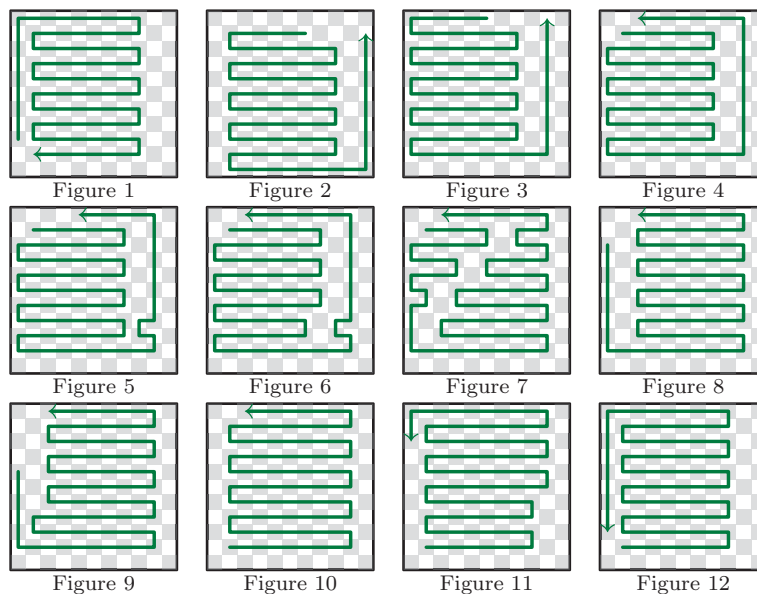
§1.3 Solution to TST 3, by Nikolai Beluhov

A *snake of length* k is an animal which occupies an ordered k-tuple $(s_1, \ldots, s_k)$ of cells in an $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \ldots, k-1$. If the snake is currently occupying $(s_1, \ldots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can *move* to occupy $(s, s_1, \ldots, s_{k-1})$ instead. The snake has *turned around* if it occupied $(s_1, s_2, \ldots, s_k)$ at the beginning, but after a finite number of moves occupies $(s_k, s_{k-1}, \ldots, s_1)$ instead.

Determine whether there exists an integer $n > 1$ such that one can place some snake of length at least $0.9n^2$ in an $n \times n$ grid which can turn around.

The answer is yes (and 0.9 is arbitrary).

First grid-based solution The following solution is due to Brian Lawrence. For illustration reasons, we give below a figure of a snake of length 89 turning around in an 11×11 square (which generalizes readily to odd n). We will see that a snake of length $(n-1)(n-2) - 1$ can turn around in an $n \times n$ square, so this certainly implies the problem.

Figure 1   Figure 2   Figure 3   Figure 4

Figure 5   Figure 6   Figure 7   Figure 8

Figure 9   Figure 10   Figure 11   Figure 12

Use the obvious coordinate system with $(1, 1)$ in the bottom left. Start with the snake as shown in Figure 1, then have it move to $(2, 1)$, $(2, n)$, $(n, n - 1)$ as in Figure 2. Then, have the snake shift to the position in Figure 3; this is possible since the snake can just walk to (n, n), then start walking to the left and then follow the route; by the time it reaches the ith row from the top its tail will have vacated by then. Once it achieves Figure 3, move the head of the snake to $(3, n)$ to achieve Figure 4.

In Figure 5 and 6, the snake begins to "deform" its loop continuously. In general, this deformation by two squares is possible in the following way. The snake walks first to $(1, n)$ then retraces the steps left by its tail, except when it reaches $(n - 1, 3)$ it makes a brief detour to $(n - 2, 3)$, $(n - 2, 4)$, $(n - 1, 4)$ and continues along its way; this gives the position in Figure 5. Then it retraces the entire loop again, except that when it reaches $(n - 4, 4)$ it turns directly down, and continues retracing its path; thus at the end of this second revolution, we arrive at Figure 6.

By repeatedly doing perturbations of two cells, we can move move all the "bumps" in the path gradually to protrude from the right; Figure 7 shows a partial application of the procedure, with the final state as shown in Figure 8.

In Figure 9, we stretch the bottom-most bump by two more cells; this shortens the "tail" by two units, which is fine. Doing this for all $(n-3)/2$ bumps arrives at the situation in Figure 10, with the snake's head at $(3, n)$. We then begin deforming the turns on the bottom-right by two steps each as in Figure 11, which visually will increase the length of the head. Doing this arrives finally at the situation in Figure 12. Thus the snake has turned around.

Second solution phrased using graph theory (Nikolai Beluhov)   Let G be any undirected graph. Consider a snake of length k lying within G, with each segment of the snake occupying one vertex, consecutive segments occupying adjacent vertices, and no two segments occupying the same vertex. One move of the snake consists of the snake's head advancing to an adjacent empty vertex and segment i advancing to the vertex of segment $i - 1$ for $i = 2, 3, \ldots, k$.

The solution proceeds in two stages. First we construct a planar graph G such that it is possible for a snake that occupies nearly all of G to turn around inside G. Then we construct a subgraph H of a grid adjacency graph such that H is isomorphic to G and H occupies nearly all of the grid.

For the first stage of the solution, we construct G as follows.

Let r and ℓ be positive integers. Start with r disjoint *main* paths $p_1, p_2, \ldots, p_r$, each of length at least ℓ, with p_i leading from A_i to B_i for $i = 1, 2, \ldots, r$. Add to those r *linking* paths, one leading from B_i to A_{i+1} for each $i = 1, 2, \ldots, r-1$, and one leading from B_r to A_1. Finally, add to those two families of *transit* paths, with one family containing one transit path joining A_1 to each of $A_2, A_3, \ldots, A_r$ and the other containing one path joining B_r to each of $B_1, B_2, \ldots, B_{r-1}$. We require that all paths specified in the construction have no interior vertices in common, with the exception of transit paths in the same family.

We claim that a snake of length $(r-1)\ell$ can turn around inside G.

To this end, let the concatenation $A_1 B_1 A_2 B_2 \ldots A_r B_r$ of all main and linking paths be the *great cycle*. We refer to $A_1 B_1 A_2 B_2 \ldots A_r B_r$ as the counterclockwise orientation of the great cycle, and to $B_r A_r B_{r-1} A_{r-1} \ldots B_1 A_1$ as its clockwise orientation.

Place the snake so that its tail is at A_1 and its body extends counterclockwise along the great cycle. Then let the snake manoeuvre as follows. (We track only the snake's head, as its movement uniquely determines the movement of the complete body of the snake.)

At phase 1, advance counterclockwise along the great cycle to B_{r-1}, take a detour along a transit path to B_r, and advance clockwise along the great cycle to A_r.

For $i = 2, 3, \ldots, r-1$, at phase i, take a detour along a transit path to A_1, advance counterclockwise along the great cycle to B_{r-i}, take a detour along a transit path to B_r, and advance clockwise along the great cycle to A_{r-i+1}.

At phase r, simply advance clockwise along the great cycle to A_1.

For the second stage of the solution, let n be a sufficiently large positive integer. Consider an $n \times n$ grid S. Number the columns of S from 1 to n from left to right, and its rows from 1 to n from bottom to top.

Let $a_1, a_2, \ldots, a_{r+1}$ be cells of S such that all of $a_1, a_2, \ldots, a_{r+1}$ lie in column 2, a_1 lies in row 2, a_{r+1} lies in row $n-1$, and $a_1, a_2, \ldots, a_{r+1}$ are approximately equally spaced. Let $b_1, b_2, \ldots, b_r$ be cells of S such that all of $b_1, b_2, \ldots, b_r$ lie in column $n-2$ and b_i lies in the row of a_{i+1} for $i = 1, 2, \ldots, r$.

Construct H as follows. For $i = 1, 2, \ldots, r$, let the main path from a_i to b_i fill up the rectangle bounded by the rows and columns of a_i and b_i nearly completely. Then every main path is of length approximately $\frac{1}{r} n^2$.

For $i = 1, 2, \ldots, r-1$, let the linking path that leads from b_i to a_{i+1} lie inside the row of b_i and a_{i+1} and let the linking path that leads from b_r to a_1 lie inside row n, column n, and row 1.

Lastly, let the union of the first family of transit paths be column 1 and let the union of the second family of transit paths be column $n-1$, with the exception of their bottommost and topmost squares.

As in the first stage of the solution, by this construction a snake of length k approximately equal to $\frac{r-1}{r} n^2$ can turn around inside an $n \times n$ grid S. When r is fixed and n tends to infinity, $\frac{k}{n^2}$ tends to $\frac{r-1}{r}$. Furthermore, when r tends to infinity, $\frac{r-1}{r}$ tends to 1. This gives the answer.