# Cutting a rectangle

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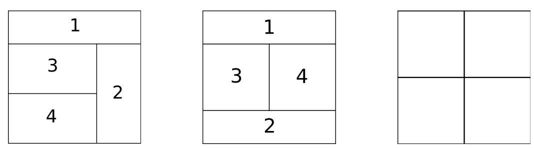
Let's consider a rectangle $R$. We define a cut of the rectangle $R$ into $n$ pieces to be a set of $n$ rectangles, each with area $\frac{1}{n}$ of the area of $R$, whose union is $R$ itself. Let $C_{R,n}$ be the set of all cuts of $R$ into $n$ pieces.

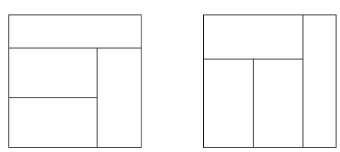
1. Let $S_{R,n} = \{x \in C_{R,n} | \ \exists$ ordering of the elements of $x : r_1, r_2, ...r_n$ such that $\forall i, \bigcup\limits_{j=i}^{n} r_j$ is a rectangle$\}$. Informally, $S_{R,n}$ is the set of cuts that can be achieved by always cutting a whole side from the biggest remaining part of $R$ with area $\frac{1}{n}$ of the area of $R$. For example:




All cuts above are from $C_{R,4}$, but only the two on the left are from $S_{R,4}$ with ordering of the elements as shown on the picture (this also corresponds to the order of cutting the pieces from the informal definition). The cut on the right is not in $S_{R,4}$ because $\bigcup\limits_{j=2}^{4} r_j$ cannot be a rectangle.

Provide a formula for $|S_{R,n}|$ with respect to $n$.

2. Let R be a square. Find the number of different cuts from $S_{R,n}$ with respect to rotations and symmetries. For example, these two cuts should be counted as one:



3. Consider $|C_{R,n}|$ with respect to $n$. Provide lower and upper bounds of $|C_{R,n}|$ (as strong as you can).

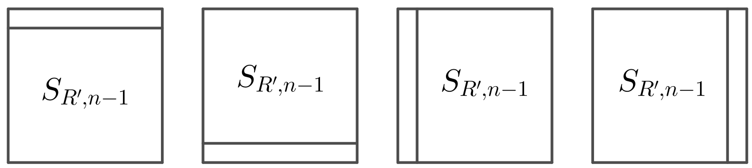
# Comments on the problems

The problems above were proposed to ITYM 2018 and were part of the final problem set. See Problem 3 in [1].

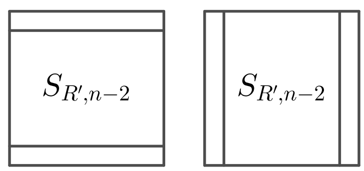
## Problem 1

This problem was also part of a programming competition. The task was to compute $|S_{R,n}|$ with respect to $1 \leq n \leq 10^{18}$ modulo $10^9 + 7$.

A recurrence relation for $|S_{R,n}|$ is provided here. There are four ways to derive a cut from $S_{R,n}$ using cuts from $S_{R',n-1}$. They are shown bellow:




But this way we will count the following twice:




Thus the recurrence relation is

$$|S_{R,n}| = 4|S_{R,n-1}| - 2|S_{R,n-2}|.$$

From the recurrence relation we can extract the general formula using the characteristic equation

$$x^2 - 4x + 2 = 0.$$

It has roots $x_{1,2} = 2 \pm \sqrt{2}$. Obviously, $S_{R,1} = 1$, and $S_{R,2} = 2$. Now we have to solve the system

$$\left| \begin{array}{l} (2+\sqrt{2})c_1 + (2-\sqrt{2})c_2 = 1 \\ (2+\sqrt{2})^2 c_1 + (2-\sqrt{2})^2 c_2 = 2. \end{array} \right.$$

We get $c_{1,2} = \dfrac{1}{2(2 \pm \sqrt{2})}$. Thus the solution is

$$|S_{R,n}| = \frac{(2+\sqrt{2})^{n-1} + (2-\sqrt{2})^{n-1}}{2}.$$

An alternative way to calculate $|S_{R,n}|$ is the following:

$$\begin{pmatrix} |S_{R,n}| \\ |S_{R,n-1}| \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

which follows directly from the recurrence formula because one multiplication with the matrix gives the next two elements. This formula is more suitable for a computation on a computer because, unlike the former, it has no problem with precision. It can be calculated by squaring in $\Theta(\log n)$ time.

## Problem 2

The standard way to approach this problem is by means of Burnside's lemma:

$$|X/G| = \frac{\sum_{g\in G} X^g}{|G|}.$$

In our case, $G$ is the group of the eight rotations and symmetries of a square, and $X$ is $S_{R,n}$. Let's find the fixed points for each element of $G$.

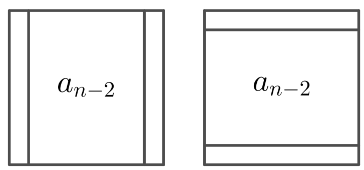
- $R_0$ is the identity. All the points in $S_{R,n}$ are fixed points of $R_0$:

$$X^{R_0} = |S_{R,n}|.$$

- $R_{90}, SR_{90}, R_{270}$, and $SR_{270}$ swap the vertical and horizontal direction. These transformations have no fixed point unless $n = 1$. Therefore,

$$X^{R_{90}} = X^{SR_{90}} = X^{R_{270}} = X^{SR_{270}} = \begin{cases} 1 & \text{if } n = 1; \\ 0 & \text{if } n > 1. \end{cases}$$

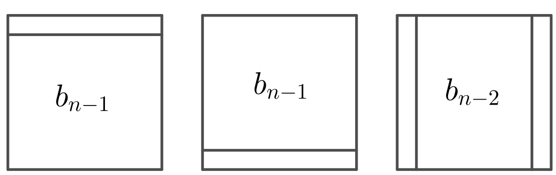
- $R_{180}$ is the rotation by $180°$.
  Let $a_n$ be the number of its fixed points in $S_{R,n}$. There are two ways to get a fixed point for a particular $n$ using a fixed point for $n-2$:



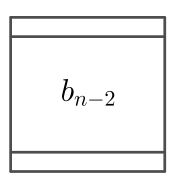

  So $a_n = 2a_{n-2}$. From $a_1 = 1$ and $a_2 = 2$ it follows that $a_n = 2^{\lfloor \frac{n}{2} \rfloor}$, i.e.

$$X^{R_{180}} = 2^{\lfloor \frac{n}{2} \rfloor}.$$

- $SR_0$ and $SR_{180}$ are the two symmetries — the former swaps left and right; the latter swaps up and down. They have the same number $b_n$ of fixed points. The fixed points of $SR_0$ look like this:




  But we count the following twice:




  Thus $b_n = 2b_{n-1} + b_{n-2} - b_{n-2} = 2b_{n-1}$. Obviously, $b_1 = 1$, $b_2 = 2$. Therefore, $b_n = 2^{n-1}$, i.e.

$$X^{SR_0} = X^{SR_{180}} = 2^{n-1}.$$

For $n > 1$, by Burnside's lemma:

$$|X/G| = \frac{|S_{R,n}| + 2^{\lfloor \frac{n}{2} \rfloor} + 2^{n-1} + 2^{n-1}}{8};$$

$$|X/G| = \frac{\frac{(2+\sqrt{2})^{n-1}+(2-\sqrt{2})^{n-1}}{2} + 2^n + 2^{\lfloor \frac{n}{2} \rfloor}}{8};$$

$$|X/G| = \frac{(2+\sqrt{2})^{n-1} + (2-\sqrt{2})^{n-1} + 2^{n+1} + 2^{\lfloor \frac{n+2}{2} \rfloor}}{16}.$$

**Alternative solution (without Burnside's lemma)**

We can derive the following relations by dynamic programming. The number of different elements of $S_{R,n}$ with respect to rotations and symmetries is $f(n,1,1)$ where $f(n,v,h)$ is the number of cuts of a rectangle into $n$ pieces such that the first piece is horizontal; $n$ is a positive integer, $v \in \{0;1\}$, $h \in \{0;1\}$; $v = 0 \iff f$ distinguishes between the upper and lower sides of the rectangle; $h = 0 \iff f$ distinguishes between the left and right sides of the rectangle.

Initial values: $f(1,v,h) = 1$. Recurrence relations for $n > 1$:

$$f(n,1,1) = 1 + \sum_{i=1}^{n-2} \left\lceil \frac{i}{2} \right\rceil f(n-i,1,0) + \sum_{\substack{i=2 \\ i \text{ – even}}}^{n-2} f(n-i,1,1);$$

$$f(n,1,0) = 1 + \sum_{i=1}^{n-2} \left\lceil \frac{i}{2} \right\rceil f(n-i,0,0) + \sum_{\substack{i=2 \\ i \text{ – even}}}^{n-2} f(n-i,0,1);$$

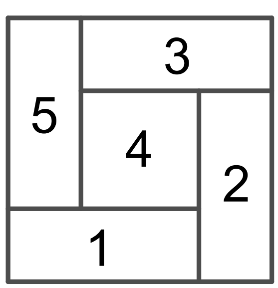
$$f(n,0,1) = 1 + \sum_{i=1}^{n-2} (i+1) f(n-i,1,0);$$

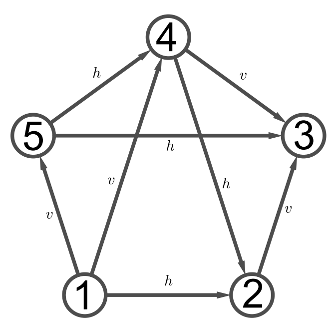
$$f(n,0,0) = 1 + \sum_{i=1}^{n-2} (i+1) f(n-i,0,0).$$

The computational complexity of this method is $\Theta(n^2)$.

## Problem 3

The first seven elements of the sequence $(|C_{R,n}|)_{n=1}^{\infty}$ can be found in OEIS [3]. Even proving that $C_{R,n}$ is finite can be a bit difficult. Actually, it is reasonable to use the number of rooted planar maps as an upper bound. Let's construct a rooted graph $G$ for each cut from $C_{R,n}$. Let the vertices of $G$ be the rectangles of the cut. Let the edges of $G$ connect neighbouring rectangles, i.e. rectangles that have a common boundary of positive length. Each edge of $G$ is marked as either vertical ($v$) or horizontal ($h$); it points upward or to the right. We choose the bottom left rectangle to be the root of $G$.



The paper [2] contains a neat proof that there cannot be more than one cut with the same graph. Since these graphs are equivalent to rooted planar maps, again from [2], we have:

$$|C_{R,n}| \leq \frac{2(2n)!\,3^n}{n!(n+2)!}.$$

From $S_{R,n} \subseteq C_{R,n}$ it follows that

$$|S_{R,n}| \leq |C_{R,n}|.$$

The rate of growth of the quantity $|C_{R,n}|$ is easily deduced. Indeed, we have $\lim\limits_{n\to\infty} \sqrt[n]{|S_{R,n}|} = 2 + \sqrt{2}$ and $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{2(2n)!\,3^n}{n!(n+2)!}} = 12$ due to Stirling's formula. Consequently,

$$1 < 2 + \sqrt{2} \leq \liminf_{n\to\infty} \sqrt[n]{|C_{R,n}|} \leq \limsup_{n\to\infty} \sqrt[n]{|C_{R,n}|} \leq 12,$$

which means that $|C_{R,n}|$ grows exponentially.

## References


[1] *ITYM 2018 Problems.*

[2] R. Häggkvist, P.-O. Lindberg, B. Lindström,
*Dissecting a square into rectangles of equal area,*
`https://www.sciencedirect.com/science/article/pii/0012365X83901036`.

[3] OEIS, `https://oeis.org/A189243`.