# 1D Piecewise Polynomials. Finite Element Method for 1D Boundary Value Problems. Additional Problems.

**Problem 1.** Consider the space $V_h$ of continuous piecewise quadratic polynomials. Define a nodal basis for this space. Derive the linear algebraic system for the coefficients of the $L_2$-projection of a given function $f$ onto $V_h$. How do you assemble the global mass matrix and load vector? Use your results to approximate the function $\sin x$ in the interval $I := [0, \pi/2]$ for $h = \pi/6$ and $h = \pi/12$. Derive an a priori error estimate for the obtained approximation.

**Problem 2.** Consider the problem

$$\begin{aligned} &-u'' + u = f, \quad x \in (0,1), \\ &u(0) = u(1) = 0. \end{aligned}$$

Derive the corresponding variational formulation. Choose a suitable finite element space $V_h$. Formulate the Ritz–Galerkin problem. Derive the discrete system of equations. Derive a priori error estimates in energy norm and in $H^1$-norm.

**Problem 3.** Compute the stiffness matrix to the problem

$$\begin{aligned} &-u'' = f, \quad x \in (0,1), \\ &u'(0) = u'(1) = 0. \end{aligned}$$

on a uniform mesh of $I$ with two elements. Why is the matrix singular?

**Problem 4.** Consider the problem

$$\begin{aligned} &-((1+x)u')' = 0, \quad x \in (0,1), \\ &u(0) = 0, \ u'(1) = 0. \end{aligned}$$

Divide the interval $I$ into three subintervals of equal length $h = 1/3$ and let $V_h$ be the corresponding space of piecewise linear polynomials, vanishing at $x = 0$. Use $V_h$ to formulate a finite element method. Compute the stiffness matrix and the load vector. Verify that the stiffness matrix is symmetric and positive-definite.

This problem has an analytic solution. Find it and then obtain an a priori error estimate for the finite element solution. Verify that this estimate indeed holds.