Finite Element Method–1
Additional Problems. Part 2.

2D Piecewise Polynomials. Finite Element Method for Stationary 2D Boundary Value Problems.

Problem 1. Determine the basis functions for piecewise-linear polynomials on an arbitrary triangle with corner coordinates

$$(x_1, y_1), (x_2, y_2), (x_3, y_3).$$

Problem 2. Consider the problem

$$\begin{aligned} -\Delta u + u &= f, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned}$$

(a) Write a variational formulation of this problem in a suitable space V.

(b) Write down a finite-element method (R.–G.), based on the variational problem.

(c) Derive an a priori error estimate in H^1 norm.

Problem 3. Consider the problem

$$\begin{aligned} -\nabla \cdot (\alpha \nabla u) + \beta u &= f, \ x \in \Omega, \\ u &= 0, \ x \in \Gamma_1, \\ -\alpha \nabla u &= 10, \ x \in \Gamma_2, \end{aligned}$$

where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$.

(a) Write a variational formulation of this problem in a suitable space V.

(b) Write down a finite-element method (R.–G.), based on the variational problem.

Coercivity and Continuity of the Bilinear Form. Related Inequalities.

Problem 4. Derive the following Poincaré ineaqualities:

$$\begin{aligned} \int_0^1 u^2 dx &\le \frac{1}{2} \int_0^1 u'^2 dx, \quad \forall u \in \mathcal{A} = \{u \in H^1[0,1] : u(1) = 0\}, \\ \int_0^1 u^2 dx &\le \frac{1}{4} \int_0^1 u'^2 dx, \quad \forall u \in H_0^1[0,1]. \end{aligned}$$

Problem 5. Show that for every $u \in C^1[0,1]$, the following inequalities hold:

$$\int_0^1 u^2 dx \le \frac{1}{6}\int_0^1 u'^2 dx + \left(\int_0^1 u dx\right)^2;$$

$$\max_{x\in[0,1]} |u(x)|^2 \le 2u^2(1) + 2\int_0^1 u'^2 dx;$$

$$\int_0^1 u^2(x) dx \le 2u^2(1) + 2\int_0^1 u'^2 dx;$$

$$\max_{x\in[0,1]} |u(x)|^2 \le 2\int_0^1 (u^2 + u'^2) dx.$$

Problem 6. Derive the weak formulation of the following fourth-order boundary-value problem, using a suitable test space V:

$$u^{(4)} + u = f(x),\ x \in (0,1),\ u(0) = u'(0) = u(1) = u'(1) = 0.$$

Show that the bilinear form $a(\cdot,\cdot)$ is coercive and continuous in V.

Problem 7. Consider the variational problem

$$\int_0^1 (u'v' + uv) dx + u(1)v(1) = \int_0^1 fv dx,\ \forall v \in V.$$

Derive the strong (differential) form of this problem if

- $V = \{v \in H_0^1(0,1) : v(0) = 0\}$;
- $V = H^1(0,1)$.

Show that in both cases the bilinear form is coercive and continuous in H^1. Derive a priori error estimates in H^1-norm for the FEM solutions, which are piecewise linear functions. Using Nitsche's trick, derive a priori error estimates in L_2-norm.