Original Problem

Find the sum

$$S := \sum_{V=1}^{\infty}\sum_{J=1}^{\infty}\sum_{I=1}^{\infty}\sum_{M=0}^{V}\sum_{C=1}^{J} \frac{(-1)^M\, 2^{4J+4M}\binom{V}{M}(J+M+1)!^2}{C\, I^2\,(I^4+4)^{J+M+1}\,(2J+2M+3)!}.$$

Technical Simplification

Using the usual notation of harmonic numbers $H_J = \sum_{C=1}^{J}\frac{1}{C}$, we obviously deduce

$$S = \sum_{V=1}^{\infty}\sum_{J=1}^{\infty}\sum_{I=1}^{\infty}\sum_{M=0}^{V} \frac{H_J\,(-1)^M\, 2^{4J+4M}\binom{V}{M}(J+M+1)!^2}{I^2(I^4+4)^{J+M+1}\,(2J+2M+3)!}.$$

Steps of the Solution

1. Show that for every $J \in \mathbb{N}$ the hamronic number H_J can be written as

$$H_J = \int_0^1 \frac{1-(1-t)^J}{t}\,\mathrm{d}t.$$

2. Let $q > 1$. Prove the identity

$$\sum_{J=1}^{\infty}\frac{H_J}{q^J} = \frac{q}{q-1}\cdot\ln\left(\frac{q}{q-1}\right).$$

3. Show that for every $n \in \mathbb{N}$ we have

$$\int_0^1 x^{n+1}\cdot(1-x)^{n+1}\mathrm{d}x = \frac{(n+1)!^2}{(2n+3)!}.$$

4. Show that for every $q > \frac{1}{4}$ we have

$$\sum_{J=1}^{\infty}\frac{H_J}{q^J}\sum_{M=0}^{V}(-1)^M\frac{\binom{V}{M}}{q^M}\cdot\frac{(J+M+1)!^2}{(2J+2M+3)!} = \int_0^1 x(1-x)\cdot\left(1-\frac{x(1-x)}{q}\right)^{V-1}\cdot\ln\left(\frac{q}{q-x(1-x)}\right)\mathrm{d}x.$$

5. Show that for every $q > \frac{1}{4}$ we have

$$\sum_{V=1}^{\infty}\sum_{J=1}^{\infty}\sum_{M=0}^{V}\frac{H_J\,(-1)^M\binom{V}{M}(J+M+1)!^2}{q^{J+M+1}\,(2J+2M+3)!} = \int_0^1 \ln\left(\frac{q}{q-x(1-x)}\right)\mathrm{d}x$$

and, consequently,

$$\frac{1}{2\sqrt{4q-1}}\sum_{V=1}^{\infty}\sum_{J=1}^{\infty}\sum_{M=0}^{V}\frac{H_J\,(-1)^M\binom{V}{M}(J+M+1)!^2}{q^{J+M+1}\,(2J+2M+3)!} = \frac{1}{\sqrt{4q-1}} - \arctan\left(\frac{1}{\sqrt{4q-1}}\right).$$

6. Further, set $\sqrt{4q-1} = \frac{I^2}{2}$ and utilize the last identity form the previous step. Summing over all positive integers I yields

$$\sum_{I=1}^{\infty}\sum_{V=1}^{\infty}\sum_{J=1}^{\infty}\sum_{M=0}^{V}\frac{H_J\,(-1)^M\binom{V}{M}(J+M+1)!^2}{I^2\left(\frac{I^4+4}{16}\right)^{J+M+1}(2J+2M+3)!} = \sum_{I=1}^{\infty}\left(\frac{2}{I^2} - \arctan\left(\frac{2}{I^2}\right)\right).$$

Evaluating both sums on the right-hand side and performing obvious simplifications on the left-hand side implies that

$$\sum_{V=1}^{\infty}\sum_{J=1}^{\infty}\sum_{I=1}^{\infty}\sum_{M=0}^{V}\sum_{C=1}^{J}\frac{(-1)^M\, 2^{4J+4M}\binom{V}{M}(J+M+1)!^2}{C\, I^2\,(I^4+4)^{J+M+1}\,(2J+2M+3)!} = \frac{4\pi^2-9\pi}{192}.$$