Zagier's "one-sentence proof"

If $p = 4k + 1$ is prime, then the set $S = \{(x, y, z) \in \mathbf{N}^3 : x^2 + 4yz = p\}$ (here the set $\mathbf{N}$ of all natural numbers can be taken to include 0 or to exclude 0, and in both cases, x, y and z must be positive for any $(x, y, z) \in S$, as p is an odd prime) is finite and has two involutions: an obvious one $(x, y, z) \to (x, z, y)$, whose fixed points, (x, y, y), correspond to representations of p as a sum of two squares, and a more complicated one,

$$(x,y,z) \mapsto \begin{cases} (x+2z,\ z,\ y-x-z), & \text{if } x < y-z \\ (2y-x,\ y,\ x-y+z), & \text{if } y-z < x < 2y \\ (x-2y,\ x-y+z,\ y), & \text{if } x > 2y \end{cases}$$

which has exactly one fixed point, $(1, 1, k)$. The cardinality of S has the same parity as the number of fixed points of an involution on that set. Thus, from the second involution we know that the cardinality of S is odd and therefore the number of fixed points for the first involution cannot be zero, proving the existence of fixed points for the first involution and consequently that p is a sum of two squares.

This proof, due to Zagier, is a simplification of an earlier proof by Heath-Brown, which in turn was inspired by a proof of Liouville.