Eulerian number

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In combinatorics, the **Eulerian number** $A(n, m)$, is the number of permutations of the numbers 1 to n in which exactly m elements are greater than the previous element (permutations with m "ascents"). They are the coefficients of the **Eulerian polynomials**:

$$A_n(t) = \sum_{m=0}^{n} A(n,m)\, t^m.$$

The Eulerian polynomials are defined by the exponential generating function

$$\sum_{n=0}^{\infty} A_n(t)\, \frac{x^n}{n!} = \frac{t-1}{t - \mathrm{e}^{(t-1)\,x}}.$$

The Eulerian polynomials can be computed by the recurrence

$$A_0(t) = 1,$$

$$A_n(t) = t\,(1-t)\,A'_{n-1}(t) + A_{n-1}(t)\,(1+(n-1)\,t), \quad n \geq 1.$$

An equivalent way to write this definition is to set the Eulerian polynomials inductively by

$$A_0(t) = 1,$$

$$A_n(t) = \sum_{k=0}^{n-1} \binom{n}{k} A_k(t)\,(t-1)^{n-1-k}, \quad n \geq 1.$$

Other notations for $A(n, m)$ are $E(n, m)$ and $\left\langle {n \atop m} \right\rangle$.

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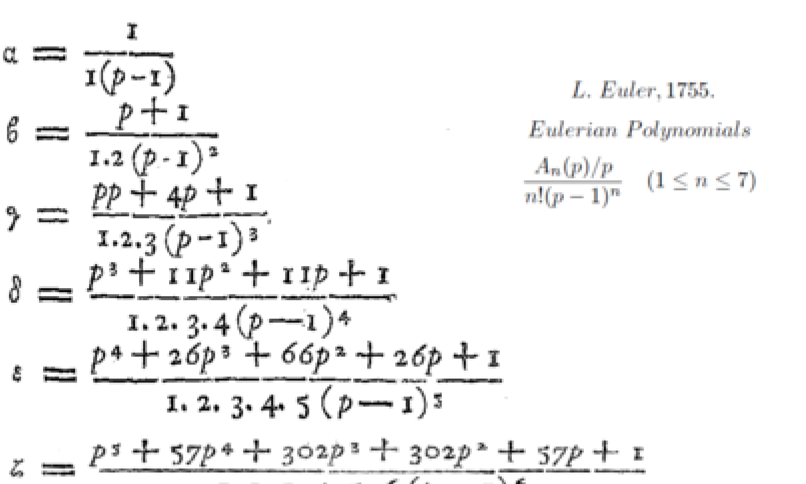

In 1755 Leonhard Euler investigated in his book *Institutiones calculi differentialis* polynomials $\alpha_1(x) = 1$, $\alpha_2(x) = x + 1$, $\alpha_3(x) = x^2 + 4x + 1$, etc. (see the facsimile). These polynomials are a shifted form of what are now called the Eulerian polynomials $A_n(x)$.

Basic properties

For a given value of $n > 0$, the index m in $A(n, m)$ can take values from 0 to $n - 1$. For fixed n there is a single permutation which has 0 ascents; this is the falling permutation $(n, n - 1, n - 2, ..., 1)$. There is also a single permutation which has $n - 1$ ascents; this is the rising permutation $(1, 2, 3, ..., n)$. Therefore $A(n, 0)$ and $A(n, n - 1)$ are 1 for all values of n.

Reversing a permutation with m ascents creates another permutation in which there are $n - m - 1$ ascents. Therefore $A(n, m) = A(n, n - m - 1)$.

Values of $A(n, m)$ can be calculated "by hand" for small values of n and m. For example

n	*m*	Permutations	*A*(*n*, *m*)
1	0	(1)	$A(1,0) = 1$
2	0	(2, 1)	$A(2,0) = 1$
	1	(1, **2**)	$A(2,1) = 1$
3	0	(3, 2, 1)	$A(3,0) = 1$
	1	(1, **3**, 2) (2, 1, **3**) (2, **3**, 1) (3, 1, **2**)	$A(3,1) = 4$
	2	(1, **2**, **3**)	$A(3,2) = 1$

For larger values of n, $A(n, m)$ can be calculated using the recursive formula

$$A(n,m) = (n-m)A(n-1,m-1) + (m+1)A(n-1,m).$$

For example

$$A(4,1) = (4-1)A(3,0) + (1+1)A(3,1) = 3 \times 1 + 2 \times 4 = 11.$$

Values of $A(n, m)$ (sequence A008292 in the OEIS) for $0 \leq n \leq 9$ are:

n \ *m*	0	1	2	3	4	5	6	7	8
1	1								
2	1	1							
3	1	4	1						
4	1	11	11	1					
5	1	26	66	26	1				
6	1	57	302	302	57	1			
7	1	120	1191	2416	1191	120	1		
8	1	247	4293	15619	15619	4293	247	1	
9	1	502	14608	88234	156190	88234	14608	502	1

The above triangular array is called the **Euler triangle** or **Euler's triangle**, and it shares some common characteristics with Pascal's triangle. The sum of row n is the factorial $n!$.

Explicit formula

An explicit formula for $A(n, m)$ is

$$A(n,m) = \sum_{k=0}^{m+1} (-1)^k \binom{n+1}{k} (m+1-k)^n.$$

Summation properties

It is clear from the combinatorial definition that the sum of the Eulerian numbers for a fixed value of n is the total number of permutations of the numbers 1 to n, so

$$\sum_{m=0}^{n-1} A(n,m) = n! \text{ for } n \geq 1.$$

The alternating sum of the Eulerian numbers for a fixed value of n is related to the Bernoulli number B_{n+1}

$$\sum_{m=0}^{n-1} (-1)^m A(n,m) = \frac{2^{n+1}(2^{n+1}-1)B_{n+1}}{n+1} \text{ for } n \geq 1.$$

Other summation properties of the Eulerian numbers are:

$$\sum_{m=0}^{n-1} (-1)^m \frac{A(n,m)}{\binom{n-1}{m}} = 0 \text{ for } n \geq 2,$$

$$\sum_{m=0}^{n-1} (-1)^m \frac{A(n,m)}{\binom{n}{m}} = (n+1)B_n \text{ for } n \geq 2,$$

where B_n is the n^{th} Bernoulli number.

Identities

The Eulerian numbers are involved in the generating function for the sequence of n^{th} powers,

$$\sum_{k=0}^{\infty} k^n x^k = \frac{\sum_{m=0}^{n-1} A(n,m) x^{m+1}}{(1-x)^{n+1}}$$

for $n \geq 0$. This assumes that $0^0 = 0$ and $A(0,0) = 1$ (since there is one permutation of no elements, and it has no ascents).

Worpitzky's identity expresses x^n as the linear combination of Eulerian numbers with binomial coefficients:

$$x^n = \sum_{m=0}^{n-1} A(n,m) \binom{x+m}{n}.$$

It follows from Worpitzky's identity that

$$\sum_{k=1}^{N} k^n = \sum_{m=0}^{n-1} A(n,m) \binom{N+1+m}{n+1}.$$

Another interesting identity is

$$\frac{e}{1-ex} = \sum_{n=0}^{\infty} \frac{A_n(x)}{n!(1-x)^{n+1}}.$$

The numerator on the right-hand side is the Eulerian polynomial.

Eulerian numbers of the second kind

The permutations of the multiset $\{1, 1, 2, 2, \cdots, n, n\}$ which have the property that for each k, all the numbers appearing between the two occurrences of k in the permutation are greater than k are counted by the double factorial number $(2n-1)!!$. The Eulerian number of the second kind, denoted $\left\langle\!\left\langle {n \atop m} \right\rangle\!\right\rangle$, counts the number of all such permutations that have exactly m ascents. For instance, for $n = 3$ there are 15 such permutations, 1 with no ascents, 8 with a single ascent, and 6 with two ascents:

332211,
221133, 221331, 223311, 233211, 113322, 133221, 331122, 331221,
112233, 122133, 112332, 123321, 133122, 122331.

The Eulerian numbers of the second kind satisfy the recurrence relation, that follows directly from the above definition:

$$\left\langle\!\!\left\langle {n \atop m} \right\rangle\!\!\right\rangle = (2n-m-1)\left\langle\!\!\left\langle {n-1 \atop m-1} \right\rangle\!\!\right\rangle + (m+1)\left\langle\!\!\left\langle {n-1 \atop m} \right\rangle\!\!\right\rangle,$$

with initial condition for $n = 0$, expressed in Iverson bracket notation:

$$\left\langle\!\!\left\langle {0 \atop m} \right\rangle\!\!\right\rangle = [m = 0].$$

Correspondingly, the Eulerian polynomial of second kind, here denoted P_n (no standard notation exists for them) are

$$P_n(x) := \sum_{m=0}^{n} \left\langle\!\!\left\langle {n \atop m} \right\rangle\!\!\right\rangle x^m$$

and the above recurrence relations are translated into a recurrence relation for the sequence $P_n(x)$:

$$P_{n+1}(x) = (2nx+1)P_n(x) - x(x-1)P_n'(x)$$

with initial condition

$$P_0(x) = 1.$$

The latter recurrence may be written in a somehow more compact form by means of an integrating factor:

$$(x-1)^{-2n-2} P_{n+1}(x) = \left(x(1-x)^{-2n-1} P_n(x)\right)'$$

so that the rational function

$$u_n(x) := (x-1)^{-2n} P_n(x)$$

satisfies a simple autonomous recurrence:

$$u_{n+1} = \left(\frac{x}{1-x} u_n\right)', \quad u_0 = 1,$$

whence one obtains the Eulerian polynomials as $P_n(x) = (1-x)^{2n} u_n(x)$, and the Eulerian numbers of the second kind as their coefficients.

Here are some values of the second order Eulerian numbers (sequence A008517 in the OEIS):

n \ *m*	0	1	2	3	4	5	6	7	8
1	1								
2	1	2							
3	1	8	6						
4	1	22	58	24					
5	1	52	328	444	120				
6	1	114	1452	4400	3708	720			
7	1	240	5610	32120	58140	33984	5040		
8	1	494	19950	195800	644020	785304	341136	40320	
9	1	1004	67260	1062500	5765500	12440064	11026296	3733920	362880

The sum of the n-th row, which is also the value $P_n(1)$, is then $(2n-1)!!$.

References

- Eulerus, Leonardus [Leonhard Euler] (1755). *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum [Foundations of differential calculus, with applications to finite analysis and series]*. Academia imperialis scientiarum Petropolitana; Berolini: Officina Michaelis.
- Graham, Knuth, Patashnik (1994). *Concrete Mathematics: A Foundation for Computer Science*, Second Edition. Addison-Wesley, pp. 267–272.
- Butzer, P. L.; Hauss, M. (1993). "Eulerian numbers with fractional order parameters". *Aequationes Mathematicae*. **46**: 119–142. doi:10.1007/bf01834003.
- T. K. Petersen (2015). *Eulerian Numbers* Birkhaüser. http://www.springer.com/us/book/9781493930906

External links

- Eulerian Polynomials (http://oeis.org/wiki/Eulerian_polynomials) at OEIS Wiki.
- "Eulerian Numbers". *MathPages.com*.
- Weisstein, Eric W. "Eulerian Number". *MathWorld*.
- Weisstein, Eric W. "Euler's Number Triangle". *MathWorld*.
- Weisstein, Eric W. "Worpitzky's Identity". *MathWorld*.
- Weisstein, Eric W. "Second-Order Eulerian Triangle". *MathWorld*.
- Euler-matrix (http://go.helms-net.de/math/binomial_new/01_12_Eulermatrix.pdf) (generalized rowindexes, divergent summation)