IMC 2011, Blagoevgrad, Bulgaria

Day 2, July 31, 2011

Problem 2. An alien race has three genders: male, female, and emale. A *married triple* consists of three persons, one from each gender, who all like each other. Any person is allowed to belong to at most one married triple. A special feature of this race is that feelings are always mutual — if x likes y, then y likes x.

The race is sending an expedition to colonize a planet. The expedition has n males, n females, and n emales. It is known that every expedition member likes at least k persons of each of the two other genders. The problem is to create as many married triples as possible to produce healthy offspring so the colony could grow and prosper.

Show that if $k \geq \frac{3n}{4}$, then it is always possible to create n disjoint married triples, thus marrying all of the expedition members.

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Solution 1.
First divide the the expedition into male-emale-female triples arbitrarily. Let the *unhappiness* of such a subdivision be the number of pairs of aliens that belong to the same triple but don't like each other. We shall show that if unhappiness is positive, then the unhappiness can be decreased by a simple operation. It will follow that after several steps the unhappiness will be reduced to zero, which will lead to the happy marriage of everybody.

Assume that we have an emale which doesn't like at least one member of its triple (the other cases are similar). We perform the following operation: we swap this emale with another emale, so that each of these two emales will like the members of their new triples. Thus the unhappiness related to this emales will decrease, and the other pairs that contribute to the unhappiness remain unchanged, therefore the unhappiness will be decreased.

So, it remains to prove that such an operation is always possible. Enumerate the triples with $1, 2, \dots, n$ and denote by E_i, F_i, M_i the emale, female, and male members of the ith triple, respectively. Without loss of generality we may assume that E_1 doesn't like either F_1 or M_1 or both. We have to find an index $i > 1$ such that E_i likes the couple F_1, M_1 and E_1 likes the couple F_i, M_i; then we can swap E_1 and E_i.

There are at most $n/4$ indices i for which E_1 dislikes F_i and at most $n/4$ indices for which E_1 dislikes M_i, so there are no more than $n/2$ indices i for which E_1 dislikes someone from the couple M_i, F_i, and the set of these undesirable indexes includes 1. Similarly, there are no more than $n/2$ indices such that either M_1 or F_1 dislikes E_i. Since both undesirable sets of indices have at most $n/2$ elements and both contain 1, their union doesn't cover all indices, so we have some i which satisfies all conditions. Therefore we can always perform the operation that decreases unhappiness.

Solution 2. Suppose that $k \geq \frac{3n}{4}$ and let's show that it's possible to marry all of the colonists. First, we'll prove that there exists a perfect matching between M and F. We need to check the condition of Hall's marriage theorem. In other words, for $A \subset M$, let $B \subset F$ be the set of all vertices of F adjacent to at least one vertex of A. Then we need to show that $|A| \leq |B|$. Let us assume the contrary, that is $|A| > |B|$. Clearly, $|B| \geq k$ if A is not empty. Let's consider any $f \in F \setminus B$. Then f is not adjacent to any vertex in A, therefore, f has degree in M not more than $n - |A| < n - |B| \leq n - k \leq \frac{n}{4}$, a contradiction.

Let's now construct a new bipartite graph, say H. The set of its vertices is $P \cup E$, where P is the set of pairs male–female from the perfect matching we just found. We will have an edge from $(m, f) = p \in P$ to $e \in E$ for each 3-cycle (m, f, e) of the graph G, where $(m, f) \in P$ and $e \in E$. Notice that the degree of each vertex of P in H is then at least $2k - n$.

What remains is to show that H satisfies the condition of Hall's marriage theorem and hence has a perfect matching. Assume, on the contrary, that the following happens. There is $A \subset P$ and $B \subset E$ such that $|A| = l$, $|B| < l$, and B is the set of all vertices of E adjacent to at least one vertex of A. Since the degree of each vertex of P is at least $2k - n$, we have $2k - n \leq |B| < l$. On the other hand, let $e \in E \setminus B$. Then for each pair $(m, f) = p \in P$, at most one of the pairs (e, m) and (e, f) is joined by an edge and hence the degree of e in G is at most $|M \setminus A| + |F \setminus A| + |A| = 2(n - l) + l = 2n - l$. But the degree of any vertex of G is $2k$ and thus we get $2k \leq 2n - l$, that is, $l \leq 2n - 2k$.

Finally, $2k - n < l \leq 2n - 2k$ implies that $k < \frac{3n}{4}$. This contradiction concludes the solution.