A Gentle Introduction to Analytic Combinatorics

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Contents

1 Introduction

1.1 General aim

- Study combinatorial structures in a simple, unified and automatic way.
- Do exact (with formal, symbolic methods) and asymptotic (with $\mathbb{C}$-analytic methods) counting.
- Examples of combinatorial structures: integers, words, permutations, trees, functional graphs.

1.2 Catalan numbers, by hand

Let's begin with one of the most famous objects in combinatorics. The approach presented here is the typical approach one would use to find the enumeration of combinatorial objects from a recurrence, as it would be described for instance in Wilf's popular textbook [4, §1].

Consider C_n the number of binary trees of size n (i.e. with n internal nodes). A simple exhaustive study leads to the first terms $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14, \ldots$

A classical way of counting those numbers is to find a recurrence. A binary tree of size $n+1$ is composed of a root and two subtrees: its left child is a binary tree of size k, its right child is a binary tree of size $n-k$, and the choice of the integer k is in the set $\{0, 1, \ldots, n\}$. So, it is possible to write the recurrence scheme

$$C_{n+1} = \sum_{k=0}^{n} C_k \, C_{n-k} \, .$$

The hint is now to use a *generating function*: $C(z) = \sum_{n \geq 0} C_n \, z^n$, where the variable z is just some parameter. The sequence $(C_n)_{n \geq 0}$ is now encoded as the function $C(z)$. From the previous equation, we multiply each side by the monomial z^{n+1}, and then make the sum for $n = 0, 1, \ldots$

$$\sum_{n \geq 0} C_{n+1} z^{n+1} = \sum_{n \geq 0} \sum_{k=0}^{n} C_k \, C_{n-k} \, z^{n+1} \, ,$$

which can be re-written

$$\sum_{n \geq 1} C_n z^n = z \sum_{n \geq 0} \sum_{k=0}^{n} \left(C_k \, z^k \right) \, \left(C_{n-k} \, z^{n-k} \right)$$

Now, using the generating function $C(z)$, we find the classical equation

$$C(z) - 1 = z \, C(z)^2$$

Solving this second order equation, and using the initial condition $C_0 = 1$ (which translates into $C(0) = 1$), the solution is

$$\boxed{C(z) = \frac{1-\sqrt{1-4z}}{2z}\,.}$$

Finding the exact coefficients C_n is done by the formal power series expansion of $C(z)$. We use the classical Newton's generalised binomial theorem

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}x^k + \ldots\,,$$

and find

$$C(z) = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} z^n\,.$$

So we conclude saying the number of binary trees of size n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. And if we want an asymptotic formula of C_n, we use the classical Stirling formula $n! \sim \sqrt{2\pi n}\, e^{-n}\, n^n$, and find

$$\boxed{C_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n\, n^{-3/2}}{\sqrt{\pi}}\,.}$$

This course's aim is to directly derive the framed results—the exact and asymptotic enumeration—from a symbolic specification of the combinatorial objects. In our current case, a binary tree can be symbolically specified as being: either a single leaf, or a node with a pair of binary trees (the left and right children), thus

$$\mathcal{B} = \bullet \text{ or } (\bullet, \mathcal{B}, \mathcal{B})$$

which of course bears a striking resemblance with the functional equation satisfied by the generating function, $C(z) = 1 + zC(z)C(z)$.

2 Unlabelled objects

This section summarizes the main aspects of the first chapter of the reference book [2, §I].

2.1 Basic definitions: combinatorial classes, generating functions

Definition 1. A *combinatorial class* $\mathcal{A}$ (sometimes simply a *class*) is a finite or denumerable set on which is defined a *size* function, $|\cdot| : \mathcal{A} \to \mathbb{Z}_{\geqslant 0}$, such that for every size there is only a *finite* number of elements, that is

$$\forall n \in \mathbb{Z}_{\geqslant 0},\ a_n := |\{x \in \mathcal{A} \mid |A| = n\}| < \infty.$$

Remark. Following the common usage (as formalized in Flajolet and Sedgewick's reference text [2]), we will always denote combinatorial classes using upper-case calligraphic letters such as $\mathcal{A}$, subclasses containing only elements of a given size as $\mathcal{A}_n$, and the counting sequences using the lower-case roman type, a_n.

As the definition suggests, for a given combinatorial class, there may be several different valid size functions. A well-known example in combinatorics is that of planar[1] binary trees: we can for instance enumerate them according to the number of internal nodes, the number of external nodes (also called *leaves*), or by counting both.

On the other hand, a trivial measure of size that would *not* be valid would be to count the number of children of the root (either 0, 1, or 2) as we would then have an infinite number of trees of "size" 1 and 2.

Definition 2. Let $\mathcal{A}$ be a combinatorial class, and let $(a_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ be its *counting sequence*. We call $A(z)$ the *ordinary generating function* (or OGF) associated with $\mathcal{A}$,

$$A(z) := \sum_{n=0}^{\infty} a_n z^n.$$

In some cases, it is also sometimes convenient to consider the equivalent definition of generating function as the sum over the objects of combinatorial class $\mathcal{A}$

$$A(z) := \sum_{\alpha\in\mathcal{A}} z^{|\alpha|}.$$

Exercise 1. Show these two definitions are equivalent.

The generating function is a traditional object in combinatorics. But where it is usually considered as a formal object, algebraically manipulated, while analytic combinatorics shows that there is considerable power in instead considering them as analytic objects.

Once given a generating function, our main goal will be to *extract its coefficients*. Let $f(z)$ be a generating function, we use the notation $[z^n]$ to note the coefficient of the variable z^n,

$$[z^n]f(z) = [z^n]\left(\sum_{i=0}^{\infty} f_i z^{\boldsymbol{i}}\right) = f_n.$$

Here are some elementary but very fundamental operations on coefficients, which also will be revisited later on.

- **Scaling:** $[z^n]f(\lambda z) = \lambda^n [z^n]f(z)$, as

$$[z^n]f(\lambda z) = [z^n]\left(\sum_{i=0}^{\infty} f_i(\lambda z)^i\right) = [z^n]\left(\sum_{i=0}^{\infty}(f_i\lambda^i)z^i\right) = \lambda^n[z^n]f(z).$$

[1] The term *planar* is here used to express that a combinatorial structure is embedded in the plane; in the case of binary trees, that means that we distinguish a left and a right child.

Combinatorial class	Counting sequence	OGF
Words on $\{0,1\}^\infty$	2^n	$W(z) = \dfrac{1}{1-2z}$
Integer compositions	2^{n-1}	$I(z) = \dfrac{1-z}{1-2z}$
Binary trees (counting internal nodes)	$\dfrac{1}{n+1}\dbinom{2n}{n}$	$B(z) = \dfrac{1-\sqrt{1-4z}}{2z}$
Permutations	$n!$	$P(z) = \sum_{n=0}^{\infty} n! z^n$

Table 1. Some standard combinatorial classes, their enumeration sequences, and their ordinary generating functions (OGFs). Note permutations do not have an analytic ordinary generating function, i.e., the radius of convergence of $P(z)$ is 0.

- **Right shifting:** $[z^n]z^k f(z) = [z^{n-k}]f(z)$, because

$$[z^n]z^k f(z) = [z^n]\left(\sum_{i=0}^{\infty} f_i z^{i+k}\right) = [z^n]\left(\sum_{i=k}^{\infty} f_{i-k} z^i\right) = [z^{n-k}]f(z).$$

2.2 The symbolic method

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be combinatorial classes with the respective ordinary generating functions $A(z)$, $B(z)$ and $C(z)$. The symbolic method is the observation that some symbolic operations can *directly be translated* to ordinary generating functions.

2.2.1 Elementary constructions

The base elements are neutral objects, noted ε, which have no size and are thus translated as 1, and atomic objects, noted $\mathcal{Z}$ and translated to OGFs as the variable z. In addition, we can distinguish however many kinds of neutral objects, for instance ε_1, ε_2, etc., which will all translate to 1, and however many kinds of atomic objects, which may translate either to the same variable z, or to some other variable z_1, z_2, etc. depending on whether it is important to distinguish the type of atom it contributes to.

Disjoint union. We write $\mathcal{A} = \mathcal{B} + \mathcal{C}$, if class $\mathcal{A}$ is defined as the disjoint union of $\mathcal{B}$ and $\mathcal{C}$: that is $\mathcal{A}$ contains all objects from $\mathcal{B}$ and $\mathcal{C}$, and objects keep their original sizes. Because the union is disjoint, there is no overlap in the enumeration, and this translates to the generating functions as

$$A(z) = B(z) + C(z).$$

Indeed, using the combinatorial definition of OGFs, since objects from $\mathcal{A}$ are either from $\mathcal{B}$ or $\mathcal{C}$,

$$A(z) = \sum_{\alpha\in\mathcal{A}} z^{|\alpha|} = \sum_{\alpha\in\mathcal{B}} z^{|\alpha|} + \sum_{\alpha\in\mathcal{C}} z^{|\alpha|} = B(z) + C(z).$$

Remark. Although we speak of "disjoint union", in practice, we never concern ourselves on whether the combinatorial classes are disjoint; instead we consider we are doing the union of unique copies of each class (for instance, imagine that $\mathcal{A} = \mathcal{B} + \mathcal{B}$ means that $\mathcal{A}$ is composed of either elements of $\mathcal{B}$ that are colored pink or purple—thus twice as many elements).

Cartesian product. We write $\mathcal{A} = \mathcal{B} \times \mathcal{C}$, if class $\mathcal{A}$ is defined as all ordered pairs, $\alpha = (\beta, \gamma) \in \mathcal{A}$ where the first element is from $\beta \in \mathcal{B}$ and the second from $\gamma \in \mathcal{C}$. The size function on $\mathcal{A}$ is then defined as $|\alpha| = |\beta| + |\gamma|$, thus

$$A(z) = B(z) \cdot C(z)$$

since

$$A(z) = \sum_{\alpha\in\mathcal{A}} z^{|\alpha|} = \sum_{\beta\in\mathcal{B}}\sum_{\gamma\in\mathcal{C}} z^{|\beta|+|\gamma|} = \left(\sum_{\alpha\in\mathcal{B}} z^{|\alpha|}\right) \cdot \left(\sum_{\alpha\in\mathcal{C}} z^{|\alpha|}\right) = B(z) \cdot C(z).$$

Remark. The size for Cartesian products is here the sum of the sizes of each object of a pair, and accordingly we say that we are dealing with *additive* combinatorial structures. Other rules for the Cartesian product are possible, for instance that the size of a pair be the *product* of each component; we would then be dealing with *multiplicative* combinatorial structures enumerated by *Dirichlet generating functions* (DGF),

$$D(s) = \sum_{n\geqslant 1} \frac{d_n}{n^s}.$$

These combinatorial structures are intimately tied to number theory, and in particular Riemann's zeta function features prominently as it is the DGF for the unit sequence (much like the quasi-inverse in additive combinatorics).

Sequence. We write $\mathcal{A} = \textsc{Seq}\,(\mathcal{B})$, if $\mathcal{A}$ is defined as all ordered sequences (of any size, including zero) of objects from $\mathcal{B}$,

$$\mathcal{A} := \{\varepsilon\} + \mathcal{B} + \mathcal{B} \times \mathcal{B} + \mathcal{B} \times \mathcal{B} \times \mathcal{B} + \ldots$$

In other words, we have

$$\mathcal{A} := \{(\beta_1, \ldots, \beta_\ell) \mid \ell \geqslant 0,\ \beta_j \in \mathcal{B}\}.$$

Observe in order for $\mathcal{A}$ to be a well-defined class, it is necessary that $b_0 = 0$ (i.e. that there be no object in $\mathcal{B}$ with size zero), as then $\mathcal{A}$ would contain an infinity of objects of any given size. The translation to OGFs is

$$A(z) = \sum_{k=0}^{\infty} (B(z))^k = \frac{1}{1 - B(z)}\ .$$

This operation is often referred to as the *quasi-inverse*.

Structure	*OGF*
$\{\varepsilon\}$	1
$\{\mathcal{Z}\}$	z
$\mathcal{A}+\mathcal{B}$	$A(z)+B(z)$
$\mathcal{A}\times\mathcal{B}$	$A(z)\cdot B(z)$
$\textsc{Seq}\,(\mathcal{A})$	$\dfrac{1}{1-A(z)}$

Table 2. Small dictionary of unlabelled combinatorial classes

Recursive classes. Finally we mention that, under certain conditions, combinatorial classes may be defined recursively, to allow for instance for the definition of branching structures. We will not go into the technical detail of these conditions (see [2, §I.2.3]), except to say that the general idea is that:

1. for every class there should be at least one terminal symbol (an atom or a neutral element);
2. a system should not allow for a same symbol to be expanded twice without increasing the size.

Example 1. This second point can be illustrated using a common mistake when specifying unary-binary trees (sometimes called Motzkin trees because they are in bijection with Motzkin paths, much like standard binary trees are in bijection with Dyck paths). If we define the class of unary-binary tree as

$$\overline{\mathcal{U}} = \mathcal{Z} + \overline{\mathcal{U}} + \overline{\mathcal{U}}^2$$

that is, we define a tree as either a leaf, or an unary internal node or a binary internal node and *we count the leaves*, then the recursion is not well-founded, and there are two ways to see this.

Combinatorically, the problem is that since unary nodes (in particular) do not affect the size of a tree, it is possible to obtain an infinity of trees of the same size, simply by taking any unary-binary tree and increasing *ad infinitum* the number of unary nodes—without changing the size. We were able to get away with counting leaves in binary trees because binary nodes affect the number of leaves (in other words, there is a direct correspondance between the number of internal nodes and external nodes).

Analytically, the problem is simply that the functional equation

$$\overline{U}(z) = z + \overline{U}(z) + \overline{U}(z)^2$$

does not admit any positive real solution. From now on, we count all the nodes.

2.2.2 Some direct examples

Example 2. Binary words on the alphabet $\{0,1\}$
A word is a finite sequence of 0 and 1.

$\mathcal{W} = \text{SEQ}(\{0\} + \{1\})$

$$W(z) = \frac{1}{1-(z+z)} \qquad \text{and} \qquad [z^n]W(z) = 2^n$$

Example 3. Number F_n of different ways to cover the segment [0,n] with bricks of size 1 and 2

Let a be an atomic class of size 1 and b an atomic class of size 2. Then, $\mathcal{F} = \text{SEQ}(a + b)$.

$$F(z) = \frac{1}{1-(z+z^2)} = 1 + z + 2z^2 + 3z^3 + 5z^4 + \ldots$$

We identify it as the Fibonacci sequence F_n. The recurrence $F_{n+2} = F_{n+1} + F_n$ is directly linked to the equation $z^2 - z - 1 = 0$.

Example 4. Integer composition [2, §I.3]

The composition of an integer n is the sequence $x_1, x_2, \ldots, x_k$ such that $n = x_1 + x_2 + \ldots + x_k$, with $x_i \geq 1$.

An integer x is an atomic class of size x, represented by the OGF z^x. The class $\mathcal{I}$ of integers has the OGF $I(z) = z + z^2 + z^3 + \ldots = \frac{z}{1-z}$.

The class of compositions of integers $\mathcal{C}$ is described by $\mathcal{C} = \text{SEQ}(\mathcal{I})$. So,

$$C(z) = \frac{1}{1-I(z)} = \frac{1}{1-\frac{z}{1-z}} = \frac{1}{1-2z} - \frac{z}{1-2z}$$

$$C_n = [z^n]C(z) = [z^n]\frac{1}{1-2z} - [z^n]\frac{z}{1-2z} = 2^n - 2^{n-1} = 2^{n-1}$$

Remark. For each example (words, Fibonacci numbers, integer compositions), the exponential growth of the coefficients of the OGF is directly linked to the singularity of the generating function (a singularity of a function is a point where the function is not well defined, when it grows to infinity).

2.3 OGFs as complex objects

Until now, an OGF is simply a formal sum of monomials. Let's now consider[2] the OGF as a univariate function of the complex variable z.

$$f(z) = \sum_{n \geq 0} f_n z^n$$

When it is possible to write f as a Taylor expansion $f(z) = \sum_{n\geq 0} \tilde{f}_n (z - z_0)^n$, we say that f is analytic at the point z_0. In combinatorics, quasi all generating functions are analytic at 0. The function f has a radius of convergence R defined by

$$R = \sup\{r \text{ such that } f(z) \text{ is analytic for } |z| < r\}$$

Another way to see the radius of convergence is

$$R^{-1} = \limsup_n |f_n|^{1/n}$$

[2]This material is covered partially in [2, §IV.1 p.225] for the complex nature of the OGF, and then the exponential growth is explained in §IV.3 p.238 and in particular §IV.3.2 p.243.

It means that when n grows to infinity, we have $f_n \sim R^{-n}\theta(n)$ where $\theta(n)$ is a subexponential function of n. The definition imposes that it must exist a singularity on the circle $|z| < r$. Furthermore, a classical theorem in complex analysis (due to Pringsheim) says: If the coefficients f_n are non-negative, then there exists a singularity at the point of the real line $z = R$.

2.4 Asymptotics of the coefficients (simple case)

Lemma 1. *(Schützenberger) With our combinatorial construction (ε, $\mathcal{Z}$, +, $\times$, SEQ), all the generating functions are rational.*

Let f be an OGF. It is possible to write f as a quotient of two polynomials $A(z)$ and $B(z)$. And so, finding the singularities of f is equivalent to finding the zeros of the denominator $B(z)$. The function f has a partial fraction expansion:

$$f(z) = \text{polynomial} + \sum_{(\rho,r),B(\rho)=0} \frac{c}{(1-z/\rho)^r}$$

Finding the asymptotics of the coefficients f_n is equivalent to the study of the asymptotics of $(1-z/\rho)^{-r}$.

$$\begin{aligned}
[z^n]\frac{1}{(1-z/\rho)^r} &= \rho^{-n}[z^n](1-z)^{-r} \\
&= \rho^{-n}\binom{n+r-1}{r-1} \\
&= \rho^{-n}\frac{(n+r-1)(n+r-2)\dots(n+1)}{(r-1)!} \\
&\sim \frac{\rho^{-n}n^{r-1}}{(r-1)!}
\end{aligned}$$

Finally, f_n is a sum of terms of the form $c\,\rho^{-n}n^{r-1}$. (This is a version of Theorem VI.1 p.381 [2], when $\rho = 1$.)

Conclusive remarks

- The singularity which is the closest to the origin give the exponential growth in the asymptotics. The singularity of minimal modulus is called a *dominant singularity*.
- The subexponential term of this asymptotic is given by the multiplicity of the dominant singularity.

Example 5. Find the asymptotics of the coefficients of

$$f(z) = (1-z^2/2)^{-5}(1-z^3)^{-1}(1-2z)^{-5}(1-z-z^2)^{-1}\,.$$

Singularities: $\{\sqrt{2}, -\sqrt{2}, 1, 1/2, \phi, \overline{\phi}\}$ Dominant singularity: $z = 1/2$ Multiplicity: 5. So, $f_n = [z^n]f(z) \sim c\,2^n n^4$.

2.5 General asymptotic scheme

With more detailed complex analysis, it is possible to get the asymptotic of other generating functions (not necessarily rational). This is Theorem VI.2 p.385 [2], also seen in the special case where the singularity is $\rho = 1$—because using the property of scaling, $[z^n]f(\rho z) = \rho^n [z^n]f(z)$, we can always get back to this case.

Theorem 1. *(Subexponential asymptotic term) For $\alpha \in \mathbb{R}\backslash\{0, -1, -2, \ldots\}$, for $k \in \mathbb{N}$,*

$$[z^n]\frac{1}{(1-z)^\alpha}\log^k\left(\frac{1}{1-z}\right) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\log^k(n),$$

where Γ is the classical generalized factorial function: $\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt$.

Theorem 2. *(Transfer lemma, Th. VI.3 p.390 [2])*
If $f(z) \sim_{z\to 1} g(z)$, then $f_n \sim g_n$.
If $f(z) =_{z\to 1} O(g(z))$, then $f_n = O(g_n)$.
If $f(z) =_{z\to 1} o(g(z))$, then $f_n = o(g_n)$.

This powerful theorem expresses that it is enough to know the comparative behaviour of two functions in the neighbourhood of their smallest singularity (here assumed to be 1).

The intuition is that a function's behaviour around its singularity is extremal and dictated exactly by its singularity.

Remark. For a more detailled lemma (with all hypothesis), see FS09. Moreover, instead of having just an equivalent, it is also possible to have a more precise asymptotic expansion with several error terms.

2.6 Tree enumeration

The topic here is first covered in [2, §I.5].

2.6.1 Binary trees $\quad \mathcal{B} = \varepsilon + \mathcal{Z}\times\mathcal{B}\times\mathcal{B}$

So, $B(z) = 1 + zB(z)^2$. We solve the equation and find $B(z) = \frac{1-\sqrt{1-4z}}{2z}$.
The singularity is at $z = 1/4$, and the order is $-1/2$.
Near $z = 1/4$, we can write $B(z) \sim \ 2 - \frac{2}{(1-4z)^{-1/2}}$. So,

$$B_n \sim -2\frac{4^n n^{-3/2}}{\Gamma(-1/2)} \sim \frac{4^n\, n^{-3/2}}{\sqrt{\pi}} \qquad \left(\Gamma(-1/2) = -2\sqrt{\pi}\right)$$

2.6.2 Unary-binary trees $\quad \mathcal{U} = \mathcal{Z} + \mathcal{Z}\times\mathcal{U} + \mathcal{Z}\times\mathcal{U}\times\mathcal{U}$

$U(z) = z + zU(z) + zU(z)^2 = z\phi(U(z))$, where $\phi(t) = 1 + t + t^2$.

Exercise 2. Find the generating function, an expression for the coefficients and an asymptotic value.

2.6.3 General trees $\mathcal{A} = \mathcal{Z} \times \textsc{Seq}(\mathcal{A})$

$$A(z) = \frac{z}{1 - A(z)} \qquad \text{so,} \qquad A(z) = z + A(z)^2$$

$$A(z) = \frac{1 - \sqrt{1 - 4z}}{2} \qquad A_n \sim \frac{4^{n-1} n^{-3/2}}{\sqrt{\pi}}$$

Remark. We notice that $zB(z) = A(z)$. Then, $[z^{n-1}]B(z) = [z^n]A(z)$, and $B_{n-1} = A_n$. The bijection between binary trees and general trees is here proved thanks to the symbolic method!

2.6.4 Otter trees: the problem of symmetries

An Otter tree $\mathcal{T}$ is a rooted binary non-planar unlabelled tree. We count the leaves.

$$T(z) = z + z^2 + z^3 + 2z^4 + 3z^5 + 6z^6 + 11z^7 + \dots$$

An Otter tree is just a leaf, or it is a node with two Otter subtrees. But there is a symmetry at this node, so we put a factor 1/2 in the counting of those configurations. But with this correction, when the two subtrees are exactly the same, it is now counted just a half time. So we add the other half for those subtrees. Then,

$$T(z) = z + \frac{1}{2}T(z)^2 + \frac{1}{2}T(z^2).$$

2.6.5 Balanced 2-3 trees: an example of substitution

Balanced 2–3 trees are trees where each node is:

- a leaf,
- an internal node with two or three sons,

and all leaves are at the same distance from the root.
The combinatorial specification is:

$$\mathcal{E} = \mathcal{Z} + \mathcal{E} \circ [\{\mathcal{Z} \times \mathcal{Z}\} + \{\mathcal{Z} \times \mathcal{Z} \times \mathcal{Z}\}]$$

3 Labelled objects and exponential generating functions

We now discuss the topic of labelled objects, introduced in [2, §II.1 and 2].

As noted, for instance in Table 1, the class of permutations does not have an analytic OGF, because the coefficients $n!$ grow exponentially faster than z^n and thus the radius of convergence of the ordinary generating function is zero.

This combinatorial explosion is a common trait shared by all combinatorial classes that are *labelled*—that is, of which the atoms are endowed with a permutation of n, the size. Permutations are such a class (a permutation is a sequence of labelled atoms), as are arrangements (a subset of labelled atoms), and more complex objects such as graphs.

3.1 Definition and examples

The solution is to enumerate these objects using *exponential generating functions*, in which the coefficient is normalized by $n!$.

Definition 3. Let $\mathcal{A}$ be a *labelled* combinatorial class, and let $(a_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ be its *counting sequence*. We call $A(z)$ the *exponential generating function* (or EGF) associated with $\mathcal{A}$,

$$A(z) := \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}.$$

And with OGFs, there is also a combinatorial definition,

$$A(z) := \sum_{\alpha\in\mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!}.$$

Notice that now, extracting the coefficient leads to a factorial factor:

$$a_n = n![z^n]A(z)$$

Example 6. $\mathcal{P}$ = {Permutations}

$$P(z) = \sum_{n\geq 0} n!\frac{z^n}{n!} = \frac{1}{1-z}$$

It looks like a sequence of atoms. Indeed, a permutation can be viewed as a linear graph of size n:

$$\sigma(1) — \sigma(2) — \sigma(3) — \ \ldots\ — \sigma(n)$$

Example 7. Non-connected graph (no edge) $\mathcal{U}$. For all n, $U_n = 1$.

$$U(z) = \sum_{n\geq 0} \frac{z^n}{n!} = e^z$$

Example 8. Complete graph (all edges) $\mathcal{K}$. It is the same EGF, $K(z) = e^z$.

Example 9. Cyclic graph (with a given orientation in the plain) $\mathcal{C}$. $C_n = (n-1)!$. So,

$$C(z) = \sum_{n\geq 1}(n-1)!\frac{z^n}{n!} = \sum_{n\geq 1}\frac{z^n}{n} = \log\left(\frac{1}{1-z}\right).$$

3.2 Construction of the sum

The disjoint union is the same construction as the unlabelled case. If $\mathcal{A} = \mathcal{B} + \mathcal{C}$, then the EGF is $A(z) = B(z) + C(z)$.

3.3 Construction of the product

Starting with two labelled structures β and γ, the classical Cartesian product does not provide a well labelled structure. The set of labels of a well-labelled structure of size n is exactly the set of integers $[1, n]$.

So, from a couple (β, γ), we define a re-labelled structure (β', γ') where the labels are exactly $\{1, \ldots, |\beta| + |\gamma|\}$, and the relative order of labels of each element is preserved. We define

$$\beta \star \gamma = \{ \text{ all couples } (\beta', \gamma') \text{ well relabelled}\}$$

The class $\beta \star \gamma$ contains exactly $\binom{|\beta|+|\gamma|}{|\beta|}$ distinct elements. Then we can define the *labelled* product

$$\mathcal{A} = \mathcal{B} \star \mathcal{C} = \bigcup_{\beta \in \mathcal{B}, \gamma \in \mathcal{C}} \beta \star \gamma$$

Lemma 2. $A(z) = B(z) \cdot C(z)$

Proof.

$$\begin{aligned}
A(z) &= \sum_{\alpha \in \mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!} \\
&= \sum_{\beta \in \mathcal{B}} \sum_{\gamma \in \mathcal{C}} \sum_{\alpha \in \beta \star \gamma} \frac{z^{|\beta|+|\gamma|}}{(|\beta| + |\gamma|)!} \\
&= \sum_{\beta \in \mathcal{B}} \sum_{\gamma \in \mathcal{C}} \binom{|\beta| + |\gamma|}{|\beta|} \frac{z^{|\beta|} z^{|\gamma|}}{(|\beta| + |\gamma|)!} \\
&= \sum_{\beta \in \mathcal{B}} \sum_{\gamma \in \mathcal{C}} \frac{z^{|\beta|} z^{|\gamma|}}{|\beta|! \, |\gamma|!} \\
&= B(z) \cdot C(z)
\end{aligned}$$

□

Remark. $\mathcal{B} \star \mathcal{B} := \mathcal{B}^2$ does not contain elements (β, β): the re-labelling make the two βs different.

3.4 Construction of the sequence

Since we have the two constructions, sum and labelled product, it is possible to construct the sequence as before. For any labelled class $\mathcal{B}$ where $b_0 = 0$,

$\mathcal{A} = \textsc{Seq}\,(\mathcal{B}) = \{\alpha \text{ s.t. } \exists k \geq 0,\ \alpha = (\beta_1, \ldots, \beta_k) \text{ finite re-labelled sequence}, \beta_i \in \mathcal{B}\}$

$$\textsc{Seq}\,(B) = \{\varepsilon\} + \mathcal{B} + \mathcal{B} \star \mathcal{B} + \mathcal{B} \star \mathcal{B} \star \mathcal{B} + \ldots$$

The corresponding EGF is

$$A(z) = \sum_{k \geq 0} B(z)^k = \frac{1}{1 - B(z)}$$

Definition 4. k-component sequence: $\textsc{Seq}_k(\mathcal{A}) = \mathcal{A}^k$

3.5 Construction of the set

A k-component set:

$$\textsc{Set}_k(\mathcal{B}) := \{\text{sets with } k \text{ elements of } \mathcal{B}\}$$

This class can be viewed as an equivalence class:

$$\textsc{Set}_k(\mathcal{B}) = \frac{\textsc{Seq}_k(\mathcal{B})}{\mathfrak{R}}$$

where $\mathfrak{R}$ is the following equivalence relation:
$(\beta_1, \ldots, \beta_k)\mathfrak{R}(\beta'_1, \ldots, \beta'_k)$ iff there exists a permutation $\sigma \in \mathfrak{S}_k$ such that $\beta_{\sigma(i)} = \beta'_i$.
We notice that the ratio of cardinalities is:

$$\frac{|\textsc{Set}_k(\mathcal{B})|}{|\textsc{Seq}_k(\mathcal{B})|} = \frac{1}{k!}\,.$$

Then, we define the $\textsc{Set}$ constructor:

$$\mathcal{A} := \textsc{Set}\,(\mathcal{B}) = \bigcup_{k\geq 0} \textsc{Set}_k(\mathcal{B})\,,$$

and the corresponding EGF is

$$A(z) = \sum_{k\geq 0} \frac{1}{k!}(B\,(z))^k = \exp(B(z))\,.$$

3.6 Construction of the cycle

For any labelled class $\mathcal{B}$ with $b_0 = 0$ and $k \geq 1$, the class of k components cycle is

$$\textsc{Cyc}_k(\mathcal{B}) := \{\text{cycles with } k \text{ elements of } \mathcal{B}\}$$

This class can be viewed as an equivalence class:

$$\textsc{Cyc}_k(\mathcal{B}) = \frac{\textsc{Seq}_k(\mathcal{B})}{\mathfrak{T}}\,,$$

where $\mathfrak{T}$ is the following equivalence relation:
$(\beta_1, \ldots, \beta_k)\mathfrak{T}(\beta'_1, \ldots, \beta'_k)$ iff there exists a cyclic permutation $\tau \in \mathfrak{S}_k$ such that $\beta_{\tau(i)} = \beta'_i$.
We notice that the ratio of cardinalities is:

$$\frac{|\textsc{Cyc}_k(\mathcal{B})|}{|\textsc{Seq}_k(\mathcal{B})|} = \frac{1}{k}\,.$$

Then, we define the $\textsc{Cyc}$ constructor:

$$\mathcal{A} := \textsc{Cyc}\,(\mathcal{B}) = \bigcup_{k\geq 1} \textsc{Cyc}_k(\mathcal{B})\,,$$

and the corresponding EGF is

$$A(z) = \sum_{k\geq 1} \frac{1}{k}\,(B\,(z))^k = \log\left(\frac{1}{1 - B(z)}\right).$$

Structure	*EGF*
$\{\varepsilon\}$	1
$\{\mathcal{Z}\}$	z
$\mathcal{A}+\mathcal{B}$	$A(z)+B(z)$
$\mathcal{A}\star\mathcal{B}$	$A(z)\cdot B(z)$
$\textsc{Seq}\,(\mathcal{A})$	$\dfrac{1}{1-A(z)}$
$\textsc{Set}\,(\mathcal{A})$	$\exp(A(z))$
$\textsc{Cyc}\,(\mathcal{A})$	$\log\left(\dfrac{1}{1-A(z)}\right)$

Table 3. Small dictionary of labelled combinatorial classes

3.7 Examples of permutation classes

3.7.1 Permutations

$$P(z) = \frac{1}{1-z} = \exp\left(\log\left(\frac{1}{1-z}\right)\right)$$

So we just a this symbolic equation:

$$\mathcal{P} = \textsc{Set}\,(\textsc{Cyc}\,(\mathcal{Z}))$$

This expreses the classical decomposition of a permutation in a product of permutation with disjoint supports.

3.7.2 Involutions

An involution σ is a permutation such that $\sigma^2 = Id$. It can be viewed as a product of permutation of size 1 and 2 with disjoint support, that is a set of cycles of size 1 or 2. All permutations are defined by: $\mathcal{P} = \textsc{Set}\,(\textsc{Cyc}\,(\mathcal{Z}))$. Involutions are specified by $\mathcal{I}$ = $\textsc{Set}(\textsc{Cyc}_{\leq 2}(\mathcal{Z}))$. Then, the EGF is

$$\begin{aligned}
I(z) &= \exp\left(z+\frac{z^2}{2}\right)\\
&= \sum_{n\geq 0}\frac{1}{n!}\left(z+z^2/2\right)^n\\
&= \sum_{n\geq 0}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^k}z^{2k}z^{n-k}\\
&= \sum_{n\geq 0}\frac{z^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^k}z^{k}
\end{aligned}$$

Extracting the coefficient,

$$\begin{aligned}
[z^n]I(z) &= \frac{1}{n!}\binom{n}{0}\frac{1}{2^0} + \frac{1}{(n-1)!}\binom{n-1}{1}\frac{1}{2^1} + \ldots + \frac{1}{(n-k)!}\binom{n-k}{k}\frac{1}{2^k} + \ldots \\
&= \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{1}{(n-i)!}\binom{n-i}{i}\frac{1}{2^i}
\end{aligned}$$

Finally, the exact number of involutions of size n is $I_n = \displaystyle\sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n!}{i!\,(n-2i)!\,2^i}$.

Remark. Finding an asymptotic for those formula will be developed later (*Saddle-point* analysis).

3.7.3 Derangements

A derangement is a permutation without fixed points.

$$\mathcal{D} = \textsc{Set}\left(\textsc{Cyc}_{>1}(\mathcal{Z})\right)$$

$$D(z) = \exp\left(\frac{z^2}{2} + \frac{z^3}{3} + \ldots\right) = \exp\left(\log\left(\frac{1}{1-z}\right) - z\right) = \frac{e^{-z}}{1-z}$$

$$d_n = n![z^n]D(z) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k(n-k)! = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

Remark. The probability for a random permutation of being a derangement is:

$$\frac{d_n}{n!} = \sum_{k=0}^{n}\frac{(-1)^k}{k!} \longrightarrow_{n\to\infty} e^{-1}$$

Remark. It can be directly done by singularity analysis. The singularity of D(z) is at $z = 1$. At this point, the asymptotic expansion of D(z) is

$$D(z) \sim_{z=1} \frac{e^{-1}}{1-z}, \qquad \text{so,} \qquad d_n \sim \frac{n!}{e}.$$

4 Recursive classes. Asymptotics of trees

(Covered in I.5 and II.5 of the book.)

In the previous examples of class of trees (binary, unary-binary, general), we saw that the generating function is often of the form $A(z) = z\phi(A(z))$. This formula expresses the classical recursive definition of tree structures.

For example,

- $\phi(t) = 1 + t + t^2$ for unary-binary trees;
- $\phi(t) = 1/(1-t)$ for general trees.

Example 10. The Cayley tree is a rooted labelled non-planar tree. Its recursive definition is a node and a set of subtrees. So, $\mathcal{T} = \mathcal{Z} \star \text{SET}(\mathcal{T})$.

$$T(z) = z\exp(T(z)).$$

For Cayley trees, $\phi(t) = e^t$.

$$\boxed{\text{How to get easily exact and asymptotic formula?}}$$

4.1 Lagrange inversion

Theorem 3. *If $A(z) = z\phi(A(z))$, then the tree equation has a unique solution which satisfies:*

$$[z^n]A(z) = \frac{1}{n}[y^{n-1}]\phi(y)^n\,;$$

$$[z^n]A(z)^k = \frac{k}{n}[y^{n-k}]\phi(y)^n\,.$$

Remark. This theorem needs some analytic hypothesis on the function ϕ, which are always verified for classical tree examples.

Proof.

Lemma 3. *If $f(z) = \sum_{n\geq 0} f_n z^n$ is analytic, then we have the Cauchy formula*

$$f_n = \frac{1}{2i\pi}\oint f(z)\frac{dz}{z^{n+1}}\,.$$

If $z = \dfrac{A(z)}{\phi(A(z))} = \dfrac{y}{\phi(y)}$, then by differentiation, $dz = \dfrac{dy}{\phi(y)} - \dfrac{y\phi'(y)}{\phi(y)^2}dy\,.$
Then, the coefficient a_n can be written:

$$\begin{aligned}
[z^n]A(z) &= \frac{1}{2i\pi}\oint y\frac{\phi(y)^{n+1}}{y^{n+1}}\left(\frac{dy}{\phi(y)} - \frac{y\phi'(y)}{\phi(y)^2}dy\right)\\
&= \frac{1}{2i\pi}\oint \frac{\phi(y)^n}{y^n}dy - \frac{1}{2i\pi}\oint \frac{\phi^{n-1}\phi'}{y^{n-1}}dy\\
&= [y^{n-1}]\phi(y)^n - \frac{1}{n}[y^{n-2}](\phi(y)^n)'
\end{aligned}$$

If we write $\phi(y)^n = \sum \alpha_p y^p$, then $(\phi(y)^n)' = \sum p\alpha_p y^{p-1}$.
So, $[z^n]A(z) = \alpha_{n-1} - \frac{1}{n}(n-1)\alpha_{n-1} = \frac{1}{n}\alpha_{n-1}$.
Finally, $[z^n]A(z) = \frac{1}{n}[y^{n-1}]\phi(y)^n$. □

4.1.1 Binary trees $\mathcal{B} = \varepsilon + \mathcal{Z} \times \mathcal{B} \times \mathcal{B}$

$B(z) = 1 + zB(z)^2$ does not fit to the specification but if we set $C(z) = B(z) - 1$, then $C(z) = z(1 + C(z))^2$. Thanks to the Lagrange inversion,

$$[z^n]C(z) = \frac{1}{n}[y^{n-1}](1+y)^{2n} = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$

4.1.2 Unary-binary trees $U(z) = z(1 + U(z) + U(z)^2)$

$$u_n = [z^n]U(z) = \frac{1}{n}[y^{n-1}](1+y+y^2)^n = \frac{1}{n}\sum_{n_1+n_2+n_3=n, n_2+2n_3=n-1}\binom{n}{n_1, n_2, n_3}$$

4.1.3 Cayley trees $\mathcal{T} = \mathcal{Z} \star \textsc{Set}\,(\mathcal{T})$

The tree equation is $T(z) = ze^{T(z)}$.

$$[z^n]T(z) = \frac{1}{n}[y^{n-1}]e^{ny} = \frac{1}{n}\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}.$$

Finally, $T_n = n![z^n]T(z) = n^{n-1}$.

4.2 Asymptotics for trees: analytic inversion

The following is based on the implicit function theorem (see Prop. IV.5 p.278 and Thm VI.6 p.404).

Theorem 4. *If $Y(z) = z\phi(Y(z))$, ϕ is an analytic function with R, radius of convergence; and if it exists a unique τ, $0 < \tau < R$ such that $\phi(\tau) = \tau\phi'(\tau)$, then $Y(z)$ is analytic at $z = 0$, its radius of convergence is $\rho = 1/\phi'(\tau)$, and $Y(z)$ has an asymptotic expansion near its singularity ρ,*

$$Y(z) \sim_{z=\rho} \tau - \gamma\sqrt{1 - z/\rho}$$

where $\gamma = \sqrt{2\phi(\tau)/\phi''(\tau)}$.

4.2.1 Unary-binary trees $U(z) = z(1 + U(z) + U(z)^2)$

We need $1 + \tau + \tau^2 = \tau(1 + 2\tau)$, which implies $\tau^2 = 1$. So, $\rho = 1/3$ and $\gamma = \sqrt{3}$.
So, for z near 1/3, $U(z) \sim 1 - \sqrt{3}\sqrt{1 - 3z}$
Finally, the singularity analysis leads to the asymptotic

$$U_n \sim \frac{\sqrt{3}}{2}\frac{3^n\, n^{-3/2}}{\sqrt{\pi}}.$$

4.2.2 Cayley trees $T(z) = ze^{T(z)}$

The equation $e^{\tau} = \tau e^{\tau}$ implies $\tau = 1$. So, the radius of convergence is $\rho = e^{-1}$, and $\gamma = \sqrt{2}$. Finally,

$$T(z) \sim_{z=e^{-1}} 1 - \sqrt{2}\sqrt{1-ez}.$$

The singularity analysis implies

$$T_n = n![z^n]T(z) \sim n!\,\frac{e^n\, n^{-3/2}}{\sqrt{2\pi}}.$$

Remark. Plus, we know that $T_n = n^{n-1}$, so it is possible to re-discover the Stirling formula

$$n^{n-1} \sim n!\frac{e^n n^{-3/2}}{\sqrt{2\pi}}\,.$$

5 Other symbolic operators

5.1 Boxed product

Let us define a modified labelled product, if $\mathcal{B}$ is a class with no element of size 0, ($b_0 = 0$).

$\mathcal{A} = \mathcal{B}^{\square} \star \mathcal{C}$ is the subset of $\mathcal{B} \star \mathcal{C}$ with labels such that the smallest label is in the $\mathcal{B}$ component. The generating function of $\mathcal{A}$ is given by

$$A(z) = \int_0^z \left(\frac{d}{dt}B(t)\right) C(t)dt\,.$$

Example 11. Records in permutations, increasing binary trees.

5.2 Pointing and substitution

Those two operations are the same in labelled and unlabelled world.

Pointing It means pointing a distinguished atom.
$\mathcal{A} = \Theta\mathcal{B}$ means $\mathcal{A}_n = [1, n] \times \mathcal{B}_n$. Constructing an object of size n in $\mathcal{A}$ is choosing an object of size n in $\mathcal{B}$ and point one of the n atoms of this object. Clearly, we have $a_n = nb_n$, so

$$A(z) = z\frac{d}{dz}B(z)\,.$$

Substitution $\mathcal{A} = \mathcal{B} \circ \mathcal{C}$ means substitute every atom of $\mathcal{B}$ by elements of $\mathcal{C}$. It translates directly into $A(z) = B(C(z))$.

6 Multivariate generating functions using markers

In this course we consider a very simple extension of our combinatorial objects to allow for the analysis of special parameters in function of the size of an object. For simplicity, we will restrain ourselves to a simple type of parameter that can be expressed in terms of *markers* (see III.1 p.152), but the technique is powerful enough to consider much more advanced parameters, for instance recursive (see III.5 p.181) or extremal (see III.8 p.214).

6.1 Definitions

Definition 5. A parameter χ for a combinatorial class $\mathcal{A}$ is a function $\chi : \mathcal{A} \longrightarrow \mathbb{N}$.

Example 12. Number of letters in a word, height of a tree, number of disconnected nodes in a graph.

Definition 6. Let $\mathcal{A}$ be a class and χ be a parameter on $\mathcal{A}$. The bivariate generating function (BGF) associated to this couple $(\mathcal{A}, \chi)$ is

$$A(z,u) := \sum_{\alpha \in \mathcal{A}} z^{|\alpha|} u^{\chi(\alpha)} \text{ (unlabelled)}, \qquad A(z,u) := \sum_{\alpha \in \mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!} u^{\chi(\alpha)} \text{ (labelled)}.$$

Equivalently, we have

$$A(z,u) = \sum_{n,k \geq 0} a_{n,k} z^n u^k \text{ (unlabelled)}, \qquad A(z,u) = \sum_{n,k \geq 0} a_{n,k} \frac{z^n}{n!} u^k \text{ (labelled)},$$

where

$$a_{n,k} = |\{\alpha \in \mathcal{A} \text{ such that } |\alpha| = n,\ \chi(\alpha) = k\}|.$$

Notation $[z^n u^k] A(z,u) = a_{n,k}$ (unlabelled) and $\dfrac{a_{n,k}}{n!}$ (labelled).

Remark. When u is set to 1, we obtain the univariate OGF (or EGF):
$A(z,1) = \sum_n \sum_k a_{n,k} z^n 1^k = \sum_n a_n z^n = A(z)$.

6.2 Symbolic method

All previous symbolic constructions are preserved when we use multivariate generating functions. Now, in the specifications, we are allowed to add markers, stickers (•) on the objects.

In the unlabelled world, we still have a direct correspondence for Union, Product, Sequence. In the labelled world, we still have a direct correspondence for Union, Product, Sequence, Set, Cycle.

Example 13. (Binary words)
We want to count the number of ones in a binary word (with alphabet $\{0, 1\}$).

$\mathcal{W} = \textsc{Seq}\,(\mathcal{Z}_0 + \bullet\mathcal{Z}_1)$, so the bivariate generating function is $W(z,u) = \dfrac{1}{1-(z+uz)}$.

$$[z^n u^k]W(z,u) = [u^k][z^n](1-z(1+u))^{-1} = [u^k](1+u)^n = \binom{n}{k}.$$

This is the number of words of size n with k ones.
$W(z,1) = (1-2z)^{-1}$, so $[z^n]W(z,1) = 2^n$.
The distribution is then easy to compute:

$$\mathbb{P}_n[\text{drawing a word with } k \text{ ones}] = \frac{\binom{n}{k}}{2^n} = \frac{[z^n u^k]W(z,u)}{[z^n]W(z,1)}\,.$$

6.3 Distribution, mean, variance, moments

What is said here applies to all multivariate generating functions, even obtained with more powerful techniques than markers (see III.2 p.156).

Definition 7. (Distribution) For a class $\mathcal{A}$ and a parameter χ, we have the BGF $A(z,u)$. The distribution of the parameter χ, uniformly with respect to the size, is given by

$$\mathbb{P}_n[\chi = k] = \frac{[z^n u^k]A(z,u)}{[z^n]A(z,1)}\,.$$

Remark. We always consider that objects of the same size have the same probability to be chosen. For a class $\mathcal{A}$, we consider there is a uniform distribution over $\mathcal{A}_n$.

Definition 8. (Mean) For a class $\mathcal{A}$, a parameter χ and the associated BGF $A(z,u)$, the expected value of the parameter χ is given by

$$\mathbb{E}_n[\chi] = \frac{[z^n]\left(\frac{d}{du}A(z,u)\right)|_{u=1}}{[z^n]A(z,1)}\,.$$

Proof.

$$\begin{aligned}
\frac{[z^n]\left(\frac{d}{du}A(z,u)\right)|_{u=1}}{[z^n]A(z,1)} &= \frac{[z^n]\left(\sum_{n,k} k\, a_{n,k} z^n u^{k-1}\right)|_{u=1}}{[z^n]\sum_n a_n z^n} = \frac{[z^n]\sum_{n,k} k\, a_{n,k} z^n}{a_n} \\
&= \frac{\sum_k k\, a_{n,k}}{a_n} = \sum_k k \frac{a_{n,k}}{a_n} \\
&= \sum_k k\, \mathbb{P}_n[\chi = k] = \mathbb{E}_n(\chi)
\end{aligned}$$

$\square$

Definition 9. (Moments) For a class $\mathcal{A}$, a parameter χ and the associated BGF $A(z,u)$, the factorial moment of order r of the parameter χ is given by

$$\mathbb{E}_n[\chi(\chi-1)\ldots(\chi-r+1)] = \frac{[z^n]\left(\frac{d^r}{du^r}A(z,u)\right)|_{u=1}}{[z^n]A(z,1)}.$$

In particular, the variance is given by

$$\mathbb{V}_n(\chi) = \mathbb{E}_n[\chi(\chi-1)] + \mathbb{E}_n[\chi] - \mathbb{E}_n[\chi]^2 \,.$$

Example 14. (Binary words)
$W(z,u) = (1 - z(1+u))^{-1}$.

$$\begin{aligned} [z^n]\left(\frac{d}{du}A(z,u)\right)|_{u=1} &= [z^n]\left(\frac{z}{(1-z(1+u))^2}\right)|_{u=1} = [z^n]\frac{z}{(1-2z)^2} \\ &= [z^{n-1}]\frac{1}{(1-2z)^2} = 2^{n-1}[z^{n-1}]\frac{1}{(1-z)^2} = 2^{n-1}n \end{aligned}$$

Finally, $\mathbb{E}_n[\text{number of ones}] = \frac{2^{n-1}n}{2^n} = \frac{n}{2}$, which is hopefully the result we expected.

Example 15. (Giving back money) We have only coins of size 1, 2, and 5. The problem is to know what is the expected number of coins we receive, in general. The specification is in the unlabelled world, and giving money is just a sequence of coins of size 1, then a sequence of coins of size 2, and finally a sequence of coins of size 5. On the specification, we choose to mark the number of coins of size 2.

$$\mathcal{D} = \textsc{Seq}\left(\mathcal{Z}\right) \times \textsc{Seq}\left(\bullet\mathcal{Z}^2\right) \times \textsc{Seq}\left(\mathcal{Z}^5\right)$$

So, the corresponding generating function is:

$$D(z,u) = \frac{1}{(1-z)}\frac{1}{(1-uz^2)}\frac{1}{(1-z^5)} \,.$$

The cumulative function $C(z) := \frac{d}{du}D(z,u)|_{u=1}$ is given by

$$C(z) = \frac{z^2}{(1-z^2)^2(1-z)(1-z^5)} \,.$$

All the poles of this function are on the circle of convergence $|z| = 1$. But, the singularity $z = 1$ is the only dominant singularity because of its multiplicity (which is 4). So, the subexponential term of asymptotic is $n^{4-1} = n^3$. The constant factor is given by the asymptotic equivalent near the singularity $z = 1$,

$$C(z) \sim_{z=1} \frac{1}{(1-z)^4(1+z)^2(1+z+z^2+z^3+z^4)} \sim \frac{1}{2^2\cdot 5}\frac{n^3}{3!} \,.$$

With the same technique of singularity analysis, we find $[z^n]D(z) \sim \frac{1}{2\cdot 5}\frac{n^2}{2}$
So the expected number of coins of size 2 is $\mathbb{E}_n[\text{coins of size 2}] \sim \frac{n}{6}$.
The same analysis can be done for the expected number of coins of size 1 and 5, and we find:

$$\mathbb{E}_n[\text{coins of size 1}] \sim \frac{n}{3} \qquad , \qquad \mathbb{E}_n[\text{coins of size 5}] \sim \frac{n}{15} \,.$$

So, the expected number of coins is $\mathbb{E}_n[\text{number of coins}] \sim \frac{n}{3}(1 + 1/2 + 1/5) \sim \frac{17n}{30}$.

7 Tree statistics

Example 16. (Root degree of a rooted tree ("Cayley tree"), Ex III.12 p.179)
The aim of this problem is to find the average number of children at the root of a Cayley tree.

Specification:

$$\mathcal{T}_{\circ} = \mathcal{Z} \star \textsc{Set}\,(\bullet\mathcal{T})$$
$$\mathcal{T} = \mathcal{Z} \star \textsc{Set}\,(\mathcal{T})$$

So the generating functions satisfy

$$T(z,u) = z\exp\left(uT(z)\right)$$
$$T(z) = z\exp\left(T(z)\right)$$

The derivative is $\frac{d}{du}T(z,u) = zT(z)\exp(uT(z))$. So, for $u = 1$, we have an expression for the cumulative function

$$\frac{d}{du}T(z,u)|_{u=1} = T(z)z\exp(T(z)) = T(z)^2\,.$$

Using the Lagrange inversion, we find the coefficient of z^n:

$$[z^n]\frac{d}{du}T(z,u)|_{u=1} = [z^n]T(z)^2 = \frac{2}{n}\,[y^{n-2}]\exp(y)^n = \frac{2}{n}n^{n-2}[y^{n-2}]e^y = \frac{2}{n}\frac{n^{n-2}}{(n-2)!}\,.$$

Finally, since, $T(z,1) = T(z) = \sum_n n^{n-1}\frac{z^n}{n!}$, the expected number of children at the root is given by

$$\mathbb{E}_n[\text{children at the root}] = \frac{[z^n]\frac{d}{du}T(z,u)|_{u=1}}{[z^n]T(z)} = \frac{2n^{n-2}}{n(n-2)!}\cdot\frac{n!}{n^{n-1}} = 2\left(1-\frac{1}{n}\right).$$

Conclusion: in general, a rooted tree has 2 children at the root!

Remark 1. Note that a nice direct proof exists: in a graph $G = (V,E)$, where V is the set of vertices and E the set of edges, $\sum_{v\in V}\deg(v) = 2|E|$. Let r be the root,

$$\begin{aligned}
\mathbb{E}_n[\deg(r)] &= \sum_{v\in V}\mathbb{P}_n[v \text{ is root}]\deg(v) \\
&= \frac{1}{n}\sum_{v\in V}\deg(v) && \text{[all vertices equiprobably the root]} \\
&= \frac{2|E|}{n} && \text{[total degree formula]} \\
&= 2\left(1-\frac{1}{n}\right) && \text{[a tree has } n-1 \text{ edges].}
\end{aligned}$$

Indeed, direct methods can generally be simpler (especially for the toy examples considered in this course to illustrate our methods), but analytic combinatorics generally presents the advantage of providing a generic, “one size fits all” method to tackle combinatorial problems which can be specified.

8 Permutation statistics

We can use all the concepts previously presented (EGF, BGF, symbolic method and singularity analysis) for the study of some statistics on permutations.

8.1 Prisoners and boxes

Puzzle A hundred prisoners, each uniquely identified by a number between 1 and 100, have been sentenced to death. The director of the prison gives them a last chance. He has a cabinet with 100 drawers (numbered 1 to 100). In each, he'll place at random a card with a prisoner's number (all numbers different). Prisoners will be allowed to enter the room one after the other and open, then close again, 50 drawers of their own choosing, but will not in any way be allowed to communicate with one another afterwards. The goal of each prisoner is to locate the drawer that contains his own number. If *all* prisoners succeed, then they will all be spared; if at least one fails, they will all be executed.

There are two mathematicians among the prisoners. The first one, a pessimist, declares that their overall chances of success are only of the order of $1/2^{100} \simeq 8{\cdot}10^{-31}$. The second one, a combinatorialist, claims he has a strategy for the prisoners, which has a greater than 30% chance of success. Who is right?

Remark. This problem, described in the book in Notes II.15 p.124 and III.10 p.176, takes its origin from a paper by Gál and Miltersen on data structures [3, 5]. The optimality of the strategy was recently proven in 2006 by Curtin and Warshauer [1].

Solution The better strategy goes as follows. Each prisoner will first open the drawer which corresponds to his number. If his number is not there, he'll use the number he just found to access another drawer, then find a number there that points him to a third drawer, and so on, hoping to return to his original drawer in at most 50 trials. (The last opened drawer will then contain his number.) This strategy globally succeeds provided the initial permutation σ defined by σ_i (the number contained in drawer i) has *all* its cycles of length at most 50. The probability of the event is

$$p = [z^{100}] \exp\left(\frac{z}{1} + \frac{z^2}{2} + \cdots + \frac{z^{50}}{50}\right) = 1 - \sum_{j=51}^{100} \frac{1}{j} \simeq 0.31182\,78206.$$

Do the prisoners stand a chance against a malicious director who would not place the numbers in drawers at random? For instance, the director might organize the numbers in a cyclic permutation. [Hint: randomize the problem by renumbering the drawers according to a randomly chosen permutation.]

8.2 Average number of cycles

Recall that the class of permutations can be seen as a set of cycles: $\mathcal{P} = \mathrm{SET}(\mathrm{CYC}(\mathcal{Z}))$. We want to count the number of cycles, so the specification becomes $\mathcal{P} = \mathrm{SET}(\bullet\mathrm{CYC}(\mathcal{Z}))$. The corresponding BGF is

$$P_c(z,u) = \exp\left(u \log\left(\frac{1}{1-z}\right)\right) = (1-z)^{-u}.$$

The average number of cycles is given by

$$\mathbb{E}_n[\text{number of cycles}] = \frac{[z^n]\frac{d}{du}P_c(z,u)|_{u=1}}{[z^n]P(z)} = [z^n]\frac{d}{du}P_c(z,u)|_{u=1}.$$

$$\Omega(z) := \frac{d}{du}P_c(z,u)|_{u=1} = \log\left(\frac{1}{1-z}\right)\exp\left(u\log\left(\frac{1}{1-z}\right)\right)|_{u=1} = \frac{1}{1-z}\,\log\left(\frac{1}{1-z}\right).$$

So,

$$\begin{aligned}
\mathbb{E}_n[\text{number of cycles}] = [z^n]\Omega(z) &= [z^n]\left(\sum_i z^i\right)\left(\sum_k \frac{z^k}{k}\right) \\
&= [z^n]\sum_p z^p\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{p}\right) \\
&= \sum_{i=1}^{n}\frac{1}{i} = H_n \sim_{n\to\infty} \log(n)\,.
\end{aligned}$$

8.3 Number of cycles of size r

Let d_r be the number of cycles of size r in a permutation of size n. In the specification of a permutation, we now want to mark only the cycles of size r.

$$\mathcal{P}_{d_r} = \textsc{Set}((\textsc{Cyc}(\mathcal{Z})\backslash\{\textsc{Cyc}_r(\mathcal{Z})\}) + \{\bullet\textsc{Cyc}_r(\mathcal{Z})\})\,.$$

The corresponding BGF is

$$P_{d_r}(z,u) = \exp\left(\log\left(\frac{1}{1-z}\right) - \frac{z^r}{r} + u\frac{z^r}{r}\right) = \frac{1}{1-z}\exp\left((u-1)\frac{z^r}{r}\right).$$

$[u^kz^n]P_{d_r}(z,u) = \frac{n![u^kz^n]P_{d_r}(z,u)}{n![z^n](1-z)^{-1}}$ is the probability that a permutation of size n has exactly k cycles of size r. This function $P_{d_r}(z,u)$ has a singularity at $z=1$, so using the transfer lemma,

$$\begin{aligned}
[u^kz^n]P_{d_r}(z,u) &\sim [u^kz^n]\frac{1}{1-z}e^{-1/r}e^{u/r} \\
&\sim e^{-1/r}\left([u^k]e^{u/r}\right)\left([z^n]\frac{1}{1-z}\right) \\
&\sim \frac{1}{k!}\frac{1}{r^k}e^{-1/r}\,.
\end{aligned}$$

So, we conclude saying the number of cycles of size r in a permutation of size n follows a Poisson law of parameter $\frac{1}{r}$.

$$\mathbb{P}_n[d_r = k] \sim \frac{1}{k!}\frac{1}{r^k}e^{-1/r} \qquad \text{so,} \qquad d_r \sim \mathcal{P}\text{oisson}\left(\frac{1}{r}\right).$$

Remark. (Expected number of cycles of size r)
In order to find this quantity, we have several options. As we know d_r follows a Poisson law of parameter r^{-1} when $n\to\infty$, we can directly say that $\mathbb{E}_n(d_r)\sim r^{-1}$.

Or, we can use the asymptotic of the cumulative function $C_{d_r}(z) = \frac{d}{du}P_{d_r}(z,u)|_{u=1}$.

$$C_{d_r}(z) = \frac{1}{1-z}e^{\frac{-z^r}{r}}\frac{z^r}{r}e^{\frac{z^r}{r}} = \frac{1}{r}\frac{z^r}{1-z}\,.$$

So,

$$\mathbb{E}_n(d_r) = \frac{n![z^n]C_{d_r}(z)}{n!} = [z^n]C_{d_r}(z) = \frac{1}{r}[z^{n-r}]\frac{1}{1-z} = \frac{1}{r}\,, \qquad \text{for } r \in \{1, \dots n\}\,.$$

This expression is exact, so it is possible to conclude on the average number of cycles in a permutation:

$$\mathbb{E}_n[\text{number of cycles}] = \sum_{r=1}^{n} \mathbb{E}_n(d_r) = \sum_{r=1}^{n} \frac{1}{r} \sim_{n\to\infty} \log(n)\,.$$

9 Functional graph statistics

We define $\mathcal{M}$, the class of mappings (or functions), by

$$\mathcal{M}_n = \{f : \{1, \dots, n\} \to \{1, \dots, n\}\}.$$

We will represent a mapping of $\mathcal{M}_n$ by a graph with n vertices, and there is an edge between two vertices, from i to j, if $f(i) = j$. The class of graphs we obtain is called *functional graphs*, and they can be viewed as graphs where every vertex has outdegree 1.

Starting from a vertex x, apply several times the function f: x, $f(x)$, $f(f(x))$ ... At some point, since the domain is finite, this construction will loop back on itself. Repeating the process for all vertices, we thus construct the whole graph. It is generally composed of several connected components and each component is an oriented cycle of points (possibly reduced at only one point), and from each point of the circle, there is some tree structure which is hung. This tree structure is a rooted tree without order on its children, so it is a Cayley tree. The specification derives from this description:

$$\mathcal{M} = \textsc{Set}\,(\textsc{Cyc}\,(\mathcal{T}))$$
$$\mathcal{T} = \mathcal{Z} \star \textsc{Set}\,(\mathcal{T})$$

The corresponding generating functions are

$$M(z) = \exp\left(\log\left(\frac{1}{1-T(z)}\right)\right) = \frac{1}{1-T(z)}\,,$$
$$T(z) = z \cdot \exp(T(z)).$$

We study tree statistics on this structure of functional graphs:

1. γ_1 is the number of cycles (connected components);
2. γ_2 is the number of cyclic points (vertices of the cycles);
3. γ_3 is the number of points without antecedents (leaves of the Cayley trees).

So we will consider three bivariate generating functions, called $M_i(z, u)$ for $i = 1, 2, 3$. The goal of this study is to find the expected value of each parameter γ_i. We know the expression of the expectation:

$$\mathbb{E}_n[\gamma_i] = \frac{n![z^n]C_i(z)}{n![z^n]M(z)}\,,$$

where $C_i(z)$ is the corresponding cumulative function $C_i(z) := \frac{d}{du} M_i(z,u)|_{u=1}$. The total number of mappings m_n is known and $m_n = n^n$, so the expression of the expectations reduces to

$$\mathbb{E}_n[\gamma_i] = \frac{n!}{n^n}[z^n]C_i(z)\,.$$

9.1 Expression of the BGFs

We have to find the symbolic specification for each parameter γ_i.

Number of cycles: γ_1

$$\mathcal{M}_1 = \textsc{Set}\,(\bullet\textsc{Cyc}\,(\mathcal{T})) \qquad \text{so} \qquad M_1(z,u) = \exp\left(u \log\left(\frac{1}{1-T(z)}\right)\right)\,.$$

So,

$$C_1(z) = \frac{d}{du} M_1(z,u)|_{u=1} = \frac{1}{1-T(z)} \log\left(\frac{1}{1-T(z)}\right)\,.$$

Number of cyclic points: γ_2

$$\mathcal{M}_2 = \textsc{Set}\,(\textsc{Cyc}\,(\bullet\mathcal{T})) \qquad \text{so} \qquad M_2(z,u) = \exp\left(\log\left(\frac{1}{1-uT(z)}\right)\right)\,.$$

So,

$$C_2(z) = \frac{d}{du} M_2(z,u)|_{u=1} = \frac{T(z)}{(1-T(z))^2}\,.$$

Number of points without antecedents: γ_3
$\mathcal{M}_3 = \textsc{Set}\left(\textsc{Cyc}\left(\widehat{\mathcal{T}}\right)\right)$ where $\widehat{T}$ is the class of Cayley trees where leaves (but not root) are marked. Let $\widetilde{\mathcal{T}}$ be the class of Cayley trees where all leaves are marked. The specification is

$$\begin{aligned}
\mathcal{M}_3 &= \textsc{Set}\left(\textsc{Cyc}\left(\widehat{\mathcal{T}}\right)\right)\\
\widehat{\mathcal{T}} &= \widetilde{\mathcal{T}} \setminus \{\bullet\mathcal{Z}\} + \{\mathcal{Z}\}\\
\widetilde{\mathcal{T}} &= \left(\mathcal{Z} \star \textsc{Set}(\widetilde{\mathcal{T}}) \setminus \{\mathcal{Z}\}\right) + \{\bullet\mathcal{Z}\}
\end{aligned}$$

The corresponding bivariate generating functions are

$$\begin{aligned}
M_3(z,u) &= \exp\left(\log\left(\frac{1}{1-\widehat{T}(z,u)}\right)\right) = \frac{1}{1-\widehat{T}(z,u)},\\
\widehat{T}(z,u) &= \widetilde{T}(z,u) - uz + \mathrm{z}\,,\\
\widetilde{T}(z,u) &= z\exp\left(\widetilde{T}(z,u)\right) + (u-1)z\,.
\end{aligned}$$

So, the cumulative function can be expressed and we find

$$C_3(z) = \frac{d}{du} M_3(z,u)|_{u=1} = \frac{zT(z)}{(1-T(z))^3}\,.$$

9.2 Expected values

All three cumulative functions are expressed in terms of the tree function $T(z)$. The asymptotic behavior is dictated by this function. But we have already studied this function and its singularities (section 3.2, analytic inversion theorem for trees). We know that the dominant singularity of $T(z)$ is at $z = e^{-1}$, and near this singularity, $T(z)$ admits an asymptotic development

$$T(z) \underset{z=e^{-1}}{\sim} 1 - \sqrt{2}\sqrt{1-ez}\,.$$

Number of cycles: γ_1

$$\begin{aligned}
\mathbb{E}_n[\gamma_1] = \frac{n!}{n^n}[z^n]C_1(z) &= \frac{n!}{n^n}[z^n]\frac{1}{1-T(z)}\log\left(\frac{1}{1-T(z)}\right)\\
&\sim \frac{n!}{n^n}[z^n]\frac{1}{\sqrt{2}\sqrt{1-ez}}\log\left(\frac{1}{\sqrt{2}\sqrt{1-ez}}\right)\\
&\sim \frac{n!}{n^n}\frac{e^n}{2\sqrt{2}}[z^n]\frac{1}{(1-z)^{1/2}}\log\left(\frac{1}{1-z}\right)\\
&\sim \frac{n!}{n^n}\frac{e^n}{2\sqrt{2}}\frac{n^{-1/2}}{\Gamma(1/2)}\log(n) \sim \boxed{\frac{1}{2}\log(n)}
\end{aligned}$$

Number of cyclic points: γ_2

$$\begin{aligned}
\mathbb{E}_n[\gamma_2] = \frac{n!}{n^n}[z^n]C_2(z) &= \frac{n!}{n^n}[z^n]\frac{T(z)}{(1-T(z))^2}\\
&\sim \frac{n!}{n^n}[z^n]\frac{1}{2(1-ez)}\\
&\sim \frac{n!}{n^n}\frac{e^n}{2}[z^n]\frac{1}{(1-z)} \sim \boxed{\sqrt{\frac{\pi n}{2}}}
\end{aligned}$$

Number of points without antecedents: γ_3

$$\begin{aligned}
\mathbb{E}_n[\gamma_3] = \frac{n!}{n^n}[z^n]C_3(z) &= \frac{n!}{n^n}[z^n]\frac{zT(z)}{(1-T(z))^3}\\
&\sim \frac{n!}{n^n}[z^n]\frac{e^{-1}}{2\sqrt{2}(1-ez)^{3/2}}\\
&\sim \frac{n!}{n^n}\frac{e^n e^{-1}}{2\sqrt{2}}[z^n]\frac{1}{(1-z)^{3/2}} \quad \sim \frac{n!e^n e^{-1}}{n^n \cdot 2\sqrt{2}}\frac{n^{1/2}}{\Gamma(3/2)} \sim \boxed{\frac{n}{e}}
\end{aligned}$$

10 Probability of being a connected graph

This section will use generating functions only as formal objects. Indeed, the functions will be implicit and their radius of convergence will be 0. But it is still possible to use them.

Let $\mathcal{G}$ be the class of labelled graphs. Take $G \in \mathcal{G}$ a graph with n vertices. We have $\binom{n}{2}$ possible edges, and for each edge, we decide to have it or not. So the total number of labelled graphs with n vertices is $g_n = 2^{\binom{n}{2}}$. Then we have an expression for the generating function:

$$G(z) = \sum_{n \geq 0} 2^{\binom{n}{2}} \frac{z^n}{n!}\,.$$

Let $\mathcal{K}$ be the subclass of $\mathcal{G}$ of connected graphs. As a graph is the set of its connected components, the symbolic method provides the following equation $\mathcal{G} = \textsc{Set}\,(\mathcal{K})$. So, the direct translation into EGF is $G(z) = \exp(K(z))$. By inversion, we can formally write

$$K(z) = \log\left(1 + \sum_{n \geq 1} 2^{\binom{n}{2}} \frac{z^n}{n!}\right)\,.$$

And using the formal definition of the log, $\log(1+u) = u - u^2/2 + u^3/3 + \ldots$, we can express the number of connected graphs with n vertices as

$$\begin{aligned}
k_n = n![z^n]K(z) &= n![z^n] \log\left(1 + \sum_{n \geq 1} 2^{\binom{n}{2}} \frac{z^n}{n!}\right) \\
&= n![z^n]\left(\sum_{n \geq 1} 2^{\binom{n}{2}} \frac{z^n}{n!}\right) - \frac{1}{2} n![z^n]\left(\sum_{n \geq 1} 2^{\binom{n}{2}} \frac{z^n}{n!}\right)^2 + \frac{1}{3} n![z^n]\left(\sum_{n \geq 1} 2^{\binom{n}{2}} \frac{z^n}{n!}\right)^3 + \ldots \\
&= 2^{\binom{n}{2}} - \frac{1}{2} \sum_{n_1+n_2=n} \binom{n}{n_1,\, n_2} 2^{\binom{n_1}{2}} 2^{\binom{n_2}{2}} + \frac{1}{3} \sum_{n_1+n_2+n_3=n} \binom{n}{n_1,\, n_2,\, n_3} 2^{\binom{n_1}{2}} 2^{\binom{n_2}{2}} 2^{\binom{n_3}{2}} + \ldots
\end{aligned}$$

In these sums, there are only a few dominant terms. Indeed, the sequence $\left(2^{\binom{n}{2}}\right)_n$ increases exponentially:

$$2^{\binom{n+1}{2}} = 2^n 2^{\binom{n}{2}}\,.$$

So, in the first sum, only the first and the last term are meaningful with regard to the asymptotic (that is $n_1 = 1$ and $n_2 = n-1$, or $n_1 = n-1$ and $n_2 = 1$). The other terms and the other sums are all included into $o\left(2^{\binom{n}{2}} 2^{-n}\right)$. So,

$$k_n = 2^{\binom{n}{2}} \left(1 - 2n2^{-n} + o(2^{-n})\right)\,.$$

Finally, almost all labelled graphs of size n are connected:

$$\mathbb{P}_n[\text{a graph is connected}] = \frac{k_n}{g_n} \underset{n \to \infty}{\sim} 1 - 2n2^{-n} \underset{n \to \infty}{\longrightarrow} 1\,.$$

11 Saddle-point method

What can we say about the asymptotics of coefficients of a generating function without singularities?

Let $f(z) = \sum_{n\geq 0} f_n z^n$ be a generating function with no singularities: it means $f(z)$ is analytic at $\mathbb{C}$. The only formula we can use is the Cauchy formula for coefficients:

$$f_n = \frac{1}{2i\pi} \oint \frac{f(z)\, dz}{z^{n+1}}\,,$$

where the integral is evaluated around some contour which englobes 0. The theory says that any contour around 0 can be used. The saddle-point method relies on a good choice of contour in order to make approximation, and asymptotic expansion.

The function inside the integral is $\frac{f(z)}{z^{n+1}}$. This function has a pole at $z = 0$. Furthermore, as f is $\mathbb{C}$-analytic (or *entire* function), the ratio grows to infinity when $|z|$ tends to infinity. Let us recap the geography of the problem. The function $\frac{f(z)}{z^{n+1}}$ has a peak at $z = 0$ and has another peak around $z \to \infty$. So, between these two peaks, there exists a valley, and especially a point with a smallest height: this point is called a *saddle-point*.

Definition 10. (saddle-point) A saddle-point z_0 of a function f is a point such that $f(z_0) \neq 0$ and $f'(z_0) = 0$.

Another key to understand the saddle-point method is the fact it is easy to evaluate contour integrals of the form $\oint e^{h(z)} dz$. Indeed, for such integrals, we locate the saddle-point z_0 where $h'(z_0) = 0$, and then, around this saddle-point, we use the Taylor expansion of h

$$h(z) = h(z_0) + \frac{1}{2} h''(z_0)(z - z_0)^2 + O((z - z_0)^3)\,.$$

So, for the evaluation of the contour integral, we cut the contour into two parts: a part $\mathcal{C}_1$ really close to the saddle-point z_0, and the other part $\mathcal{C}_2$ (the rest of the circle centered at 0). For the part $\mathcal{C}_1$, we use the Taylor expansion of h, then a constant term $e^{h(z_0)}$ will be a factor, and the rest of the integral is easy to evaluate (directly related to $\int e^{-t^2} dt$). Then, it is possible to show that the integral on the part $\mathcal{C}_2$ is exponentially negligible.

Saddle-point technique Let $f(z) = \sum f_n z^n$. Let note $\exp(h(z)) = \frac{f(z)}{z^{n+1}}$. Then, find ζ_n such that $h'(\zeta_n) = 0$, that is

$$\zeta_n \frac{f'(\zeta_n)}{f(\zeta_n)} = n + 1\,.$$

Then we have an asymptotic expression for the coefficients

$$f_n \sim \frac{f(\zeta_n)}{\zeta_n^{n+1}\sqrt{2\pi h''(\zeta_n)}}\,.$$

11.1 Exponential and 1/n!

If $f(z) = e^z$, we already know that $[z^n]f(z) = \frac{1}{n!}$. The function f has no singularity so we can, as an exercise, use the saddle-point method. Let $h(z) = \log \frac{f(z)}{z^{n+1}} = z - (n+1)\log(z)$.
So, $h'(z) = 1 - \frac{n+1}{z}$, and $h''(z) = \frac{n+1}{z^2}$. $h'(\zeta_n) = 0$ implies $\zeta_n = n+1$.
So, we can deduce an asymptotic for the factorial

$$\frac{1}{n!} \sim \frac{e^{n+1}}{(n+1)^{n+1}\sqrt{2\pi/(n+1)}} .$$

Then, we put one factor $(n+1)$ inside the square root, put the factor n^n outside, and use the equivalent $(1+1/n)^n \sim e$, and we find

$$\frac{1}{n!} \sim \frac{e^n}{n^n\sqrt{2\pi n}} .$$

11.2 Number of involutions: asymptotics

Remember that the generating function of the involutions is $I(z) = \exp\left(z + \frac{z^2}{2}\right)$. This function has no singularity, so we use the saddle-point method. Let $\exp(h(z)) = I(z)/z^{n+1}$, so we have

$$h(z) = z + \frac{z^2}{2} - (n+1)\log(z) , \quad h'(z) = 1 + z - \frac{n+1}{z} , \quad h''(z) = 1 + \frac{n+1}{z^2} .$$

The derivative cancels for the roots of $z^2 + z - (n+1)$. The positive saddle-point is $-1/2 + 1/2\sqrt{1+4(n+1)}$. When n tends to infinity, it is sufficient to know the asymptotics of the saddle-point, so

$$\zeta_n \sim \sqrt{n} - 1/2 + \frac{5}{8\sqrt{n}} .$$

We obtain an expression for $[z^n]I(z)$:

$$\frac{I_n}{n!} \sim \frac{e^{h(\zeta_n)}}{\zeta_n^{n+1}\sqrt{2\pi h''(\zeta_n)}} \sim \frac{e^{n/2+\sqrt{n}-1/4}\, n^{-n/2}}{2\sqrt{\pi n}} \qquad \left(\sim \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{1}{i!\,(n-2i)!\,2^i}\right) ,$$

$$I_n \sim \frac{1}{\sqrt{2}} \exp\left(\sqrt{n} - \frac{n}{2} - \frac{1}{4}\right) n^{n/2}.$$

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