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Mathematics

Decision procedure for indefinite hypergeometric summation

(algorithm/binomial coefficient identities/closed form/symbolic computation/linear recurrences)

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ABSTRACT Given a summand a_n, we seek the "indefinite sum" $S(n)$ determined (within an additive constant) by

$$\sum_{n=1}^{m} a_n = S(m) - S(0) \qquad [0]$$

or, equivalently, by

$$a_n = S(n) - S(n-1). \qquad [1]$$

An algorithm is exhibited which, given a_n, finds those $S(n)$ with the property

$$\frac{S(n)}{S(n-1)} = \text{a rational function of } n. \qquad [2]$$

With this algorithm, we can determine, for example, the three identities

$$\sum_{n=1}^{m} \frac{\prod_{j=1}^{n-1} bj^2 + cj + d}{\prod_{j=1}^{n} bj^2 + cj + e} = \frac{1 - \prod_{j=1}^{m} \dfrac{bj^2 + cj + d}{bj^2 + cj + e}}{e - d}, \qquad [3a]$$

$$\sum_{n=1}^{m} \frac{\prod_{j=1}^{n-1} aj^3 + bj^2 + cj + d}{\prod_{j=1}^{n} aj^3 + bj^2 + cj + e} = \frac{1 - \prod_{j=1}^{m} \dfrac{aj^3 + bj^2 + cj + d}{aj^3 + bj^2 + cj + e}}{e - d}, \qquad [3b]$$

and
$$\sum_{n=1}^{m} \frac{\prod_{j=1}^{n-1} bj^2 + cj + d}{\prod_{j=1}^{n+1} bj^2 + cj + e} = \frac{\dfrac{2b}{e-d} - \dfrac{3b+c+d-e}{b+c+e} - \left(\dfrac{2b}{e-d} - \dfrac{b(2m+3)+c+d-e}{b(m+1)^2 + c(m+1) + e}\right) \prod_{j=1}^{m} \dfrac{bj^2 + cj + d}{bj^2 + cj + e}}{b^2 - c^2 + d^2 + e^2 + 2bd - 2de + 2eb} \qquad [3c]$$

and we can also conclude that

$$\sum_{n=1}^{m} \frac{\prod_{j=1}^{n-1} j^3}{\prod_{j=1}^{n+1} j^3 + 1} \qquad [3d]$$

is inexpressible as $S(m) - S(0)$, for any $S(n)$ satisfying Eq. 2.

The technique relies on a particular change of variables to reduce Eq. 1 to a system of linear equations which is consistent if Eq. 2 holds.

Method

If $S(n)/S(n-1)$ is a rational function of n, then by Eq. 1 the term ratio

$$\frac{a_n}{a_{n-1}} = \frac{S(n) - S(n-1)}{S(n-1) - S(n-2)} = \frac{\dfrac{S(n)}{S(n-1)} - 1}{1 - \dfrac{S(n-2)}{S(n-1)}} \qquad [4]$$

must also be a rational function of n. (We exclude the degenerate case where a_n is identically zero.) Express this ratio as

$$\frac{a_n}{a_{n-1}} = \frac{p_n}{p_{n-1}} \frac{q_n}{r_n}, \qquad [5]$$

where p_n, q_n, and r_n are polynomials in n subject to the following condition:

$$\gcd(q_n, r_{n+j}) = 1, \qquad [6]$$

for all non-negative integers j.

It is always possible to put a rational function in this form, for if $\gcd(q_n, r_{n+j}) = g(n)$, then this common factor can be eliminated with the change of variables

$$q_n' \leftarrow \frac{q_n}{g(n)},\; r_n' \leftarrow \frac{r_n}{g(n-j)},$$
$$p_n' \leftarrow p_n g(n) g(n-1) \ldots g(n-j+1), \qquad [6']$$

which leaves the term ratio unchanged. The values of j for which such gs exist can be readily detected as the non-negative integer roots of the resultant of q_n and r_{n+j} with respect to n.

We now write

$$S(n) = \frac{q_{n+1}}{p_n} f(n) a_n, \qquad [7]$$

where $f(n)$ is to be determined. By using Eq. 1,

$$f(n) = \frac{p_n}{q_{n+1}} \frac{S(n)}{S(n) - S(n-1)} = \frac{p_n}{q_{n+1}} \frac{1}{1 - \dfrac{S(n-1)}{S(n)}},$$

so $f(n)$ is a rational function of n whenever $S(n)/S(n-1)$ is. By substituting Eq. 7 into Eq. 1, we get

$$a_n = \frac{q_{n+1}}{p_n} f(n) a_n - \frac{q_n}{p_{n-1}} f(n-1) a_{n-1}.$$

Multiplying this through by p_n/a_n, and using Eq. 5, we have

$$p_n = q_{n+1} f(n) - r_n f(n-1), \qquad [8]$$

the functional equation for f.

THEOREM. *If* $S(n)/S(n-1)$ *is a rational function of* n, *then* $f(n)$ *is a polynomial.*

Proof: We already know that $f(n)$ is a rational function when $S(n)/S(n-1)$ is, so suppose

$$f(n) = \frac{c(n)}{d(n)} \qquad [9]$$

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where $d(n)$ is a polynomial of positive degree, and

$$\gcd(c(n), d(n)) = 1 = \gcd(c(n-1), d(n-1)). \quad [9a]$$

Then Eq. 8 can be rewritten

$$d(n)\,d(n-1)p_n = c(n)\,d(n-1)q_{n+1} - d(n)\,c(n-1)r_n. \quad [8']$$

Now let j be the largest integer such that

$$\gcd(d(n), d(n+j)) = g(n) \neq 1. \quad [10a]$$

Clearly j exists and is ≥ 0. Because j is maximal and $d(n+j)$ is a multiple of $g(n)$,

$$\gcd(d(n-1), d(n+j)) = 1 = \gcd(d(n-1), g(n)). \quad [10b]$$

Shifting n by $-j-1$ in Eq. **10a**,

$$\gcd(d(n-j-1), d(n-1)) = g(n-j-1) \neq 1. \quad [10c]$$

Shifting n by $-j$ in the left side of Eq. **10b**,

$$\gcd(d(n-j-1), d(n)) = 1 = \gcd(g(n-j-1), d(n)), \quad [10d]$$

since $d(n-j-1)$ is a multiple of $g(n-j-1)$.

Now consider dividing Eq. **8′** by the polynomials $g(n)$ and $g(n-j-1)$. By Eq. **10a**, $g(n)$ divides $d(n)$. By Eqs. **10b** and **9a**, $g(n)$ is relatively prime to $d(n-1)$ and $c(n)$. So in Eq. **8′**, $g(n)$ divides q_{n+1}, and thus $g(n-1)$ divides q_n.

Similarly, by Eq. **10c**, $g(n-j-1)$ divides $d(n-1)$, but then by Eqs. **10d** and **9a**, $g(n-j-1)$ is relatively prime to $d(n)$ and $c(n-1)$, so in Eq. **8′**, $g(n-j-1)$ must divide r_n, and thus $g(n-1)$ divides r_{n+j}.

Thus j is a non-negative integer for which q_n and r_{n+j} have the common factor $g(n-1)$, contradicting condition **6**, and thus $d(n)$ must be a constant, QED.

All that remains is to look for a polynomial $f(n)$ satisfying Eq. **8**, given p_n, q_n, and r_n. To do this, we must first find an upper bound, k, for the degree of $f(n)$. This we can facilitate by rewriting Eq. **8** as

$$p_n = (q_{n+1} - r_n)\frac{f(n) + f(n-1)}{2} + (q_{n+1} + r_n)\frac{f(n) - f(n-1)}{2} \quad [8'']$$

and observing that

$$\deg(f(n) + f(n-1)) = 1 + \deg(f(n) - f(n-1))$$

for any nonzero polynomial f, if we define $\deg(0) = -1$.

Case 1:

$$\deg(q_{n+1} + r_n) \leq \deg(q_{n+1} - r_n) = l.$$

Estimating both $f(n)$ and $f(n-1)$ in Eq. **8″** by

$$c_k n^k + O(n^{k-1}),$$

we get

$$p_n = L c_k n^{k+l} + O(n^{k+l-1}),$$

where L is a nonzero constant. Since the two sides of the equation must be of equal degree,

$$k = \deg(f(n)) = \deg(p_n) - l.$$

Case 2:

$$\deg(q_{n+1} - r_n) < \deg(q_{n+1} + r_n) = l.$$

Estimating

$$f(n) = c_k n^k + c_{k-1}n^{k-1} + O(n^{k-2})$$

and

$$f(n-1) = c_k n^k + (c_{k-1} - k c_k)n^{k-1} + O(n^{k-2})$$

in Eq. **8″** leads to

$$p_n = L(k)c_k n^{k+l-1} + O(n^{k+l-2}),$$

where $L(k)$ is linear in k and free of c_k. Let k_0 be the root of $L(k)$. Then, the largest k for which c_k can be nonzero is

$$k = \max\{k_0, \deg(p_n) - l + 1\}, \text{ if } k_0 \text{ is an integer, or}$$

$$k = \deg(p_n) - l + 1, \text{ if not.}$$

In either case, if $k < 0$, the indefinite sum $S(n)$ (satisfying Eq. **2**) does not exist; otherwise we can substitute a kth degree polynomial, with $k+1$ undetermined coefficients, for $f(n)$ in Eq. **8**. Equating like powers of n, we get a system of linear equations whose consistency is equivalent to the existence of an $S(n)$ satisfying Eq. **2** and whose solution provides $f(n)$, whereupon the indefinite sum, $S(n)$, is given by Eq. **7**.

Applications

Formulas **3a** and **3b** are just special cases of the very general result

$$\sum_{n=1}^{m} \frac{\prod_{j=1}^{n-1} f(j)}{\prod_{j=1}^{n} f(j) + c} = \frac{1 - \prod_{j=1}^{m} \frac{f(j)}{f(j)+c}}{c}, \quad [11]$$

where f is an arbitrary function; however, our decision procedure can only establish this when f is a specified rational function. Nevertheless, the relatively simple formula **3a** implies several binomial coefficient identities, e.g.,

$$\sum_{n=1}^{m} \frac{\binom{n+p}{q}}{\binom{n+r}{q+2}} = \frac{(r+1)(p-q+1)\binom{p+1}{q}}{(q+1)(p-r+1)\binom{r+1}{q+2}} - \frac{(m+r-q-1)(m+p-q+1)\binom{m+p+1}{q}}{(q+1)(p-r+1)\binom{m+r}{q+2}}$$

which arises directly from the substitutions

$$b = \frac{\binom{r+1}{q+2}}{(r+1)(p-q+1)\binom{p+1}{q}}, \; c = (r-q+p)b,$$

$$d = (p+1)(r-q-1)b, \text{ and } e = (p-q)rb.$$

In contrast, formula **3c** does not appear to have a generalization analogous to **11**, in light of the nonexistence of a closed form for **3d**. We can, however, use the decision procedure to characterize those a, b, c, d, and e for which

$$\sum_{n=1}^{m} \frac{\prod_{j=1}^{n-1} aj^3 + bj^2 + cj + d}{\prod_{j=1}^{n+1} aj^3 + bj^2 + cj + e} \quad [12]$$

does have a closed form.

First, we suppose that in Eq. **5** the term ratio,

$$\frac{a_n}{a_{n-1}} = \frac{a(n-1)^3 + b(n-1)^2 + c(n-1) + d}{a(n+1)^3 + b(n+1)^2 + c(n+1) + e} = \frac{q_n}{r_n},$$

that is, from the start there is no j violating condition **6**, so that $p_n = 1$. Then, to determine the degree of $f(n)$, we observe that

$$\deg(q_{n+1} - r_n) = 2 < \deg(q_{n+1} + r_n) = 3 = l,$$

which requires Case 2, where we find that $k = k_0 = 3$, so f will be a cubic with four undetermined coefficients. Substituting this into Eq. **8**, we get $1 =$ a fourth degree polynomial, so we have five coefficients to annihilate with our four unknowns. This requires that the augmented coefficient matrix for the system,

$$\begin{bmatrix} -d & -d & -d & e-d+c+b+a & 1 \\ -3d-c & -2d-c & e-d+b+a & 2b+3a & 0 \\ -3d-3c-b & e-d-c+a & b+3a & 3a & 0 \\ e-d-2c-2b & 2a & 2a & 0 & 0 \\ -b & a & 0 & 0 & 0 \end{bmatrix}, \qquad [13]$$

must have its determinant

$$a(e-d)(3ae - 3ad - 6ac + 2b^2 - 6a^2) = 0, \qquad [14]$$

to ensure that at least one equation is redundant. The case $a = 0$ is just the identity **3c**. The case $e = d$ leaves **13** unsatisfiable unless both a and b are zero, which leaves a very elementary telescoping identity. But when the third factor of **14** is used to eliminate any of the five parameters, the resulting system is consistent, so that the closed form for **12** becomes

$$\frac{\dfrac{2(6a^2m^3 + 6a(3a+b)m^2 + 2(6a^2+6ab+b^2)m + 6a(a+b+c+d) + b(d-e))\prod_{j=1}^{m} aj^3+bj^2+cj+d}{\prod_{j=1}^{m+1} aj^3+bj^2+cj+e} - \dfrac{2(2b^2+3a(2b+d+e)+b(d-e))}{a+b+c+e}}{(e-d)(2b(e-d+c-2a)-9a(e+d))}, \qquad [15]$$

provided

$$3ae - 3ad - 6ac + 2b^2 - 6a^2 = 0.$$

But there are cases when **12** can be summed even when **14** does not hold—namely, when there is a non-negative integer j violating condition **6**, so that the change of variables **6′** must be made before a closed form can be ruled out. Here, the case $a = 0$ is uninteresting because identity **3c** applies, regardless of j. When $a \neq 0$, we can assume that a has been normalized to 1 by dividing all the parameters by a, and the whole sum by a^2. Then the substitution

$$d = (\alpha+1)((\alpha+1)^2 - (\alpha+1)b + c)$$
$$e = (\alpha-j-1)((\alpha-j-1)^2 - (\alpha-j-1)b + c),$$

where j is a non-negative integer, will ensure a $g(n) = n + \alpha$ for the change of variables **6′**.

$$\deg(q_{n+1} - r_n) = 1 < \deg(q_{n+1} + r_n) = 2 = l,$$

and *Case* 2 determines that $k = \deg(f) = j + 4$, whereupon Eq. **8** yields $j + 5$ equations in $j + 5$ unknowns. Computer experiments have shown this system to be consistent for $j = 0, 1, 2,$ and 3, and it is reasonable to conjecture that consistency extends to all larger j, in which case any reduction of q_n and r_n by **6′** would guarantee a closed form for **12**.

But as j increases, the resulting closed forms quickly become unintelligibly large, even for $\alpha = j + 1$. When $j = 0$ and $\alpha = 1$ (the simplest case which leaves b and c unconstrained), we have

$$\sum_{n=1}^{m} \frac{\prod_{j=1}^{n-1} (j+2)(j^2+(b-2)j+c-2b+4)}{\prod_{j=1}^{n+1} j^3+bj^2+cj} = \frac{6c^2-(b^2+14b-75)c+2b^3-74b+168}{2(c+b+1)(b-3)(c-2b+4)(16c-3b^2-14b+49)}$$
$$\frac{(6m^4+(6b+30)m^3+(12b+12c+78)m^2+((2b+42)c-4b^2-10b+138)m+6c^2-(b^2+14b-75)c+2b^3-74b+168)\prod_{j=1}^{m} j^3+bj^2+cj+2c-4b+8}{(b-3)(c-2b+4)(16c-3b^2-14b+49)(m+2)\prod_{j=1}^{m+1} j^3+bj^2+cj}. \qquad [16$$

Substituting into the determinant **14** the values of a through e that we have used in **16**, we get

$$4(2b-c-4)(b^2+6(b-c)-15).$$

Since this is, in general, nonzero, we see that **16** is the "simplest" member of a family of solutions largely disjoint from sums of the form **15**.

It should be mentioned that general formulas such as **3a** and **15** seem to include special cases, such as the harmonic series, which the decision procedure would reject as unsummable. The contradiction is avoided, however, by the observation that, in such cases, the righthand sides involve division by zero.

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