ALGORITHM OF DEMOUCRON, MALGRANGE, PERTUISET

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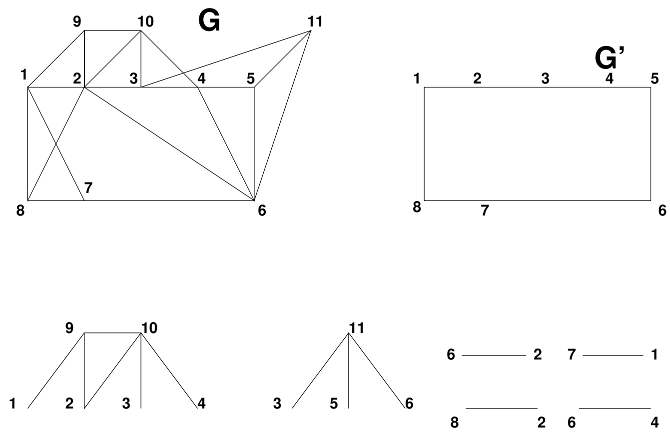
There is an algorithm from 1964 by Demoucron, Malgrange and Pertuiset [DMP64] which computes a planar embedding for a planar graph $G = (V, E)$. [MTYS94] This is an incremental algorithm as the embedding is computed step by step, where a step is the embedding of a new cycle of the graph. Assume we have already computed a embedding of a subgraph G' of G we have to look at the so called fragments.

Definition. fragments

For a subgraph $G' = (V', E')$ of $G = (V, E)$ we define a *fragment* of G (with respect to G') as a subgraph $S = (V_S, E_S)$ of G with either $E_S = \{u, v\}$ with $\{u, v\} \in E \backslash E'$ or S is a connected component of $G \backslash G'$ together with all edges in G between S and G'.

Example. 6 fragments of G with respect to G'.



A vertex v of a fragment is called *contact vertex* if v∈V'. In the above example the contact vertices of the first fragment are 1,2,3,4. For the next step of the algorithm we have to assume that our (may be) planar is biconnected. If it is not biconnected we compute planar embeddings of the biconnected components and put them together at the articulation points. For biconnected graphs we always have to different contact points of a fragment. As we are working in an incremental algorithm we assume that we have a planar embedding of G', so we can speak of faces of G'. An *admissible face* of a fragment S is a face of G' which contains all contact points of S. So if we are looking for embeddings of a fragment, the admissible faces are the only candidates. The set of all admissible faces of a fragment S is denoted by F(S). During one step of the algorithm we will embed an (elementary) path of a fragment connecting two different contact vertices. The start and the

Date: 14. December 2004.

end vertex are the only contact vertices in this path. Such a path is called *α-path*. Two different fragments S and T are called *competing* if

- $F(S) \cap F(T) \neq \emptyset$
- for each face from F(S)∩F(T) there are two α-paths L⊂S and M⊂T which cannot simultaneously embedded into this face.

Again in the above example the first and second fragment are competing as the α-paths 1-9-10-4 and 3-11-5 are not simultaneously embeddable into a common face. We are now in the position to give the algorithm

Algorithm. *planarity test with planar embedding(Demoucron, Malgrange and Pertuiset)*

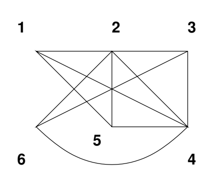
input: graph G

(1) *choose a cycle of G this is a planar graph G' together with a embedding*
(2) *compute all faces of G'*
(3) *compute F(S) = set of fragments of G with respect to G'*
(4) *if F(S)=Ø then we have G'≅G and G' has a planar embedding. end*
(5) *compute all admissible faces for all fragments*
(6) *if there is a fragment without admissible face, then the graph has no planar embedding. end*
(7) *if there is a fragment S with only one admissible face then goto 9*
(8) *choose a fragment S (more then one admissible face)*
(9) *choose a α-path from S and embed it into an admissible face of S, goto 2*

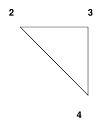
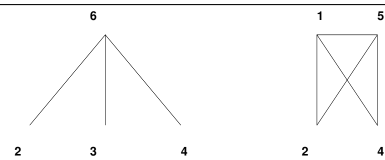
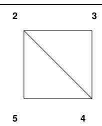
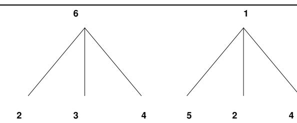
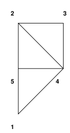
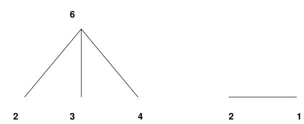
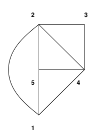
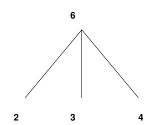
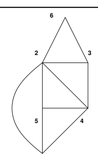
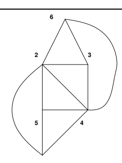
We will first do an example

Example. planar embedding

we got the following graph G



in the first step we start with an arbitrary cycle e.g. 2-3-4, the following steps are shown in the table

G'	faces	fragments	admissible faces
	F_1={∞,234} F_2={234}		$F(S_1)$=1,2 $F(S_2)$=1,2
	F_1={∞,2345} F_2={234} F_3={245}		$F(S_1)$=1,2 $F(S_2)$=1,3
	F_1={∞,12345} F_2={234} F_3={245} F_4={145}		$F(S_1)$=1,2 $F(S_2)$=1
	F_1={∞,1234} F_2={234} F_3={245} F_4={145} F_5={125}		$F(S_1)$=1,2
	F_1={∞,12346} F_2={234} F_3={245} F_4={145} F_5={125} F_6={236}		$F(S_1)$=1
			

Next step is to prove the correctness of the algorithm.

Theorem.

The algorithm of Demoucron, Malgrange and Pertuiset for the computation of a planar embedding of a graph is correct.

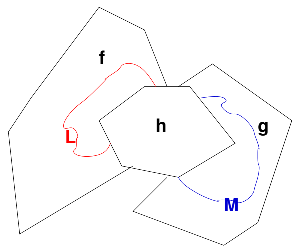
Proof. we will show this theorem in several steps. □

First step is to show that in the case of competing fragments with more than one admissible face these fragments have each exact two admissible faces and these are identical.

Lemma. *admissible faces of competing fragments*

If competing fragments S,T are such that $|F(T)| \geq 2$ *and* $|F(S)| \geq 2$ *then* $F(S) = F(T)$ *and* $|F(T)| = 2$.

Proof. Assume $F(S) \neq F(T)$ then we have at least 3 distinct faces f, g, h with $f \in F(S)$ and $g \in F(T)$ and as the fragments are competing we have a nonempty join, so $h \in F(S) \cap F(T)$. Each α-path $L \subset S$ is embeddable into face f, and each α-path $M \subset T$ is embeddable into face g, so each pair of (L, M) is embeddable outside of face h, but so it is also embeddable inside of h, which is a violation of the condition of competing fragments.



This shows $F(S) = F(T)$. To show that there are only 2 faces, we do the same construction like above for the 3 different faces $\{f, g, h\} \subset F(S)$. □

To show that we have an incremental algorithm we define a *partial embedding* G' of a planar graph G as a embedding of a subgraph G' of G which we get from a embedding of G by removing edges and vertices. So if we have a partial embedding, it can be extended to an embedding of the whole graph. As a further tool we define the *fragment graph* $S(G')$ of a fragment set. The vertices are the fragments, and they are connected by a edge if it is a pair of competing fragments.

Lemma. *fragment graph is bipartite*

if we got a partial embedding G' of G and if $F(S) \geq 2$ for all fragments then $S(G')$ is bipartite.

Proof. Assume the graph $S(G')$ is not bipartite, according to lemma**??**there is a cycle of odd length r, this cycle of odd length corresponds to a sequence $s_1 - s_2 - ... - s_r - s_1$ of competing fragments. Because of the previous lemma we have $F(s_i) = F(s_{i+1})$ and there are exact two admissible faces F_1 and F_2. So as we have a partial embedding and the only way to do a embedding of the fragments to put the paths $L_i \subset s_i$ into F_1if i is odd and into F_2 in the case of even i.(or the other way round). But as r is odd this generates a problem at the last step $s_r - s_1$ as these are both of odd index, so we would get a simultaneous embedding of paths of competing fragments. □

Last step is the following theorem

Theorem. *each step produces a partial embedding*

if G is planar, each iteration of the algorithm produces a partial embedding G'

Proof. use induction on n = the number of iterations

n=1. G' produced by the algorithm is a cycle, which can be reconstructed by removal of edges and vertices from any planar embedding of G.

Now we have a planar embedding of G'. First step of the algorithm is the computation of all admissible faces of the fragments of G'. As we are in a partial embedding of G there are admissible faces for all fragments. Otherwise there would be an α-path in a fragment connecting different faces of the partial embedding, which be a nonplanar embedding of G.

Now we have to look at step 7/8 of the algorithm.

case 1: there is a fragment S with only one admissible face. In this case there is only one possibility to embed an α-path $L \subset S$, so the algorithm produces a partial embedding G'∪L which we from the same embedding of G used for the partial embedding of G' .

case 2: all fragments have more then one admissible face. We embed the α-path $L \subset S$. In the case we have chosen the same face like in the planar embedding of G, we still have a partial embedding which we get from the planar embedding of G. This was the easy case. Assume we have chosen the 'wrong' face. In the case that there is no competing fragment, we can put the fragment at any admissible face. In the case of a competing fragment we look at the connected component of S in the fragment graph $S(G')$. It is bipartite and there are only two admissible faces $F(S) = f, g$ for all fragments in the connected component. So we know all fragments embedded in f don't compete with all the fragments in g. So if swap these two faces in the planar embedding of G, we have a new planar embedding of G which can be reduced to our partial embedding computed in the algorithm. □

Corollary. *the algorithm is correct*

(1) *if G is planar the algorithm computes a planar embedding*
(2) *if the algorithm stops because there is a fragment without an admissible face, the graph is not planar*

Proof. this is clear from the above theorem □

This algorithm is of complexity $O(|V|^2)$.

References

[DMP64] G. Demoucron, Y. Malgrange, R. Pertuiset, Graphes Planaires: Reconnaissance et Construction de Representations Planaires Topologiques, Rev. Franc. Rech. Oper. (8) 1964, 33 - 34.

[MTYS94] O. Melnikov, R. Tyshkevich, V. Yemelichev, V. Sarvanov, *Lectures on Graph Theory*, ISBN 3-411-17121-9, BI Wissenschaftsverlag, 1994.

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