Karger's algorithm

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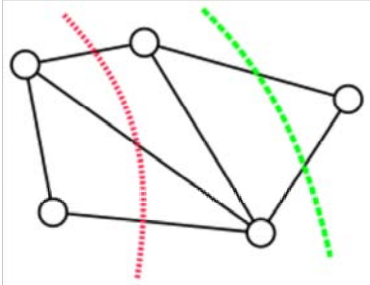
In computer science and graph theory, **Karger's algorithm** is a randomized algorithm to compute a minimum cut of a connected graph. It was invented by David Karger and first published in 1993.[1]



A graph with two cuts. The dotted line in red is a cut with three crossing edges. The dashed line in green is a min-cut of this graph, crossing only two edges.

The idea of the algorithm is based on the concept of contraction of an edge (u, v) in an undirected graph $G = (V, E)$. Informally speaking, the contraction of an edge merges the nodes u and v into one, reducing the total number of nodes of the graph by one. All other edges connecting either u or v are "reattached" to the merged node, effectively producing a multigraph. Karger's basic algorithm iteratively contracts randomly chosen edges until only two nodes remain; those nodes represent a cut in the original graph. By iterating this basic algorithm a sufficient number of times, a minimum cut can be found with high probability.

Contents

The global minimum cut problem

A *cut* (S, T) in an undirected graph $G = (V, E)$ is a partition of the vertices V into two non-empty, disjoint sets $S \cup T = V$. The *cutset* of a cut consists of the edges $\{ uv \in E \colon u \in S, v \in T \}$ between the two parts. The *size* (or *weight*) of a cut in an unweighted graph is the cardinality of the cutset, i.e., the number of edges between the two parts,

$$w(S,T) = |\{ uv \in E \colon u \in S, v \in T \}| \, .$$

There are $2^{|V|}$ ways of choosing for each vertex whether it belongs to S or to T, but two of these choices make S or T empty and do not give rise to cuts. Among the remaining choices, swapping the roles of S and T does not change the cut, so each cut is counted twice; therefore, there are $2^{|V|-1} - 1$ distinct cuts. The *minimum cut problem* is to find a cut of smallest size among these cuts.

For weighted graphs with positive edge weights $w \colon E \rightarrow \mathbf{R}^+$ the weight of the cut is the sum of the weights of edges between vertices in each part

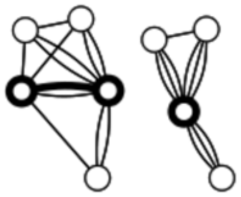
$$w(S,T) = \sum_{uv \in E \colon u \in S, v \in T} w(uv) \, ,$$

which agrees with the unweighted definition for $w = 1$.

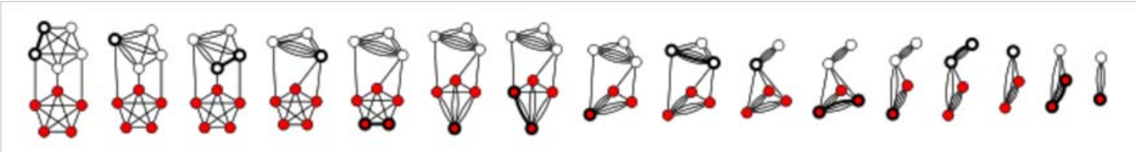
A cut is sometimes called a “global cut” to distinguish it from an “s-t cut” for a given pair of vertices, which has the additional requirement that $s \in S$ and $t \in T$. Every global cut is an s-t cut for some $s, t \in V$. Thus, the minimum cut problem can be solved in polynomial time by iterating over all choices of $s, t \in V$ and solving the resulting minimum s-t cut problem using the max-flow min-cut theorem and a polynomial time algorithm for maximum flow, such as the Ford–Fulkerson algorithm, though this approach is not optimal. There is a deterministic algorithm for the minimum cut problem with running time $O(mn + n^2 \log n)$.[2]

Contraction algorithm

The fundamental operation of Karger's algorithm is a form of edge contraction. The result of contracting the edge $e = \{u, v\}$ is new node uv. Every edge $\{w, u\}$ or $\{w, v\}$ for $w \notin \{u, v\}$ to the endpoints of the contracted edge is replaced by an edge $\{w, uv\}$ to the new node. Finally, the contracted nodes u and v with all their incident edges are removed. In particular, the resulting graph contains no self-loops. The result of contracting edge e is denoted G/e.



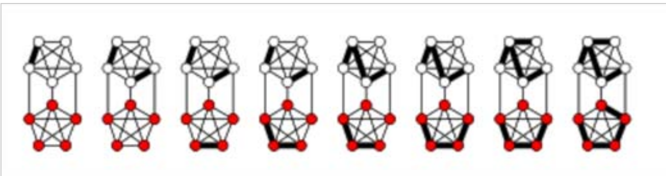
The contraction algorithm repeatedly contracts random edges in the graph, until only two nodes remain, at which point there is only a single cut.



Successful run of Karger's algorithm on a 10-vertex graph. The minimum cut has size 3.

procedure contract($G = (V, E)$):
while $|V| > 2$
    choose $e \in E$ uniformly at random
    $G \leftarrow G/e$
return the only cut in G

When the graph is represented using adjacency lists or an adjacency matrix, a single edge contraction operation can be implemented with a linear number of updates to the data structure, for a total running time of $O(|V|^2)$. Alternatively, the procedure can be viewed as an execution of Kruskal's algorithm for constructing the minimum spanning tree in a graph where the edges have weights $w(e_i) = \pi(i)$ according to a random permutation π. Removing the heaviest edge of this tree results in two components that describe a cut. In this way, the contraction procedure can be implemented like Kruskal's algorithm in time $O(|E| \log |E|)$.



The random edge choices in Karger's algorithm correspond to an execution of Kruskal's algorithm on a graph with random edge ranks until only two components remain.

The best known implementations use $O(|E|)$ time and space, or $O(|E| \log |E|)$ time and $O(|V|)$ space, respectively.[1]

Success probability of the contraction algorithm

In a graph $G = (V, E)$ with $n = |V|$ vertices, the contraction algorithm returns a minimum cut with polynomially small probability $\binom{n}{2}^{-1}$. Every graph has $2^{n-1} - 1$ cuts,[3] among which at most $\binom{n}{2}$ can be minimum cuts. Therefore, the success probability for this algorithm is much better than the probability for picking a cut at random, which is at most $\binom{n}{2}/(2^{n-1} - 1)$

For instance, the cycle graph on n vertices has exactly $\binom{n}{2}$ minimum cuts, given by every choice of 2 edges. The contraction procedure finds each of these with equal probability.

To establish the bound on the success probability in general, let C denote the edges of a specific minimum cut of size k. The contraction algorithm returns C if none of the random edges belongs to the cutset of C. In particular, the first edge contraction avoids C, which happens with probability $1 - k/|E|$. The minimum degree of G is at least k (otherwise a minimum degree vertex would induce a smaller cut), so $|E| \geq nk/2$. Thus, the probability that the contraction algorithm picks an edge from C is

$$\frac{k}{|E|} \leq \frac{k}{nk/2} = \frac{2}{n}.$$

The probability p_n that the contraction algorithm on an n-vertex graph avoids C satisfies the recurrence $p_n \geq (1 - \frac{2}{n})p_{n-1}$, with $p_2 = 1$, which can be expanded as

$$p_n \geq \prod_{i=0}^{n-3}\Bigl(1-\frac{2}{n-i}\Bigr) = \prod_{i=0}^{n-3}\frac{n-i-2}{n-i} = \frac{n-2}{n}\cdot\frac{n-3}{n-1}\cdot\frac{n-4}{n-2}\cdots\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3} = \binom{n}{2}^{-1}.$$

Repeating the contraction algorithm

By repeating the contraction algorithm $T = \binom{n}{2}\ln n$ times with independent random choices and returning the smallest cut, the probability of not finding a minimum cut is

$$\left[1 - \binom{n}{2}^{-1}\right]^T \leq \frac{1}{e^{\ln n}} = \frac{1}{n}.$$

10 repetitions of the contraction procedure. The 5th repetition finds the minimum cut of size 3.

The total running time for T repetitions for a graph with n vertices and m edges is $O(Tm) = O(n^2 m \log n)$.

Karger–Stein algorithm

An extension of Karger's algorithm due to David Karger and Clifford Stein achieves an order of magnitude improvement.[4]

The basic idea is to perform the contraction procedure until the graph reaches t vertices.

procedure contract($G = (V, E)$, t):
while $|V| > t$
    choose $e \in E$ uniformly at random
    $G \leftarrow G/e$
return G

The probability $p_{n,t}$ that this contraction procedure avoids a specific cut C in an n-vertex graph is

$$p_{n,t} \geq \prod_{i=0}^{n-t-1}\Bigl(1 - \frac{2}{n-i}\Bigr) = \binom{t}{2}\Big/\binom{n}{2}.$$

This expression is $\Omega(t^2/n^2)$ becomes less than $\frac{1}{2}$ around $t = \lceil 1 + n/\sqrt{2}\rceil$. In particular, the probability that an edge from C is contracted grows towards the end. This motivates the idea of switching to a slower algorithm after a certain number of contraction steps.

procedure fastmincut($G = (V, E)$):
if $|V| \leq 6$:
    return mincut(V)
else:
    $t \leftarrow \lceil 1 + |V|/\sqrt{2}\rceil$
    $G_1 \leftarrow$ contract(G, t)
    $G_2 \leftarrow$ contract(G, t)
    return min {fastmincut(G_1), fastmincut(G_2)}

Analysis

The probability $P(n)$ the algorithm finds a specific cutset C is given by the recurrence relation

$$P(n) = 1 - \left(1 - \frac{1}{2}P\left(\left\lceil 1 + \frac{n}{\sqrt{2}}\right\rceil\right)\right)^2$$

with solution $P(n) = O\left(\frac{1}{\log n}\right)$. The running time of fastmincut satisfies

$$T(n) = 2T\left(\left\lceil 1 + \frac{n}{\sqrt{2}}\right\rceil\right) + O(n^2)$$

with solution $T(n) = O(n^2 \log n)$. To achieve error probability $O(1/n)$, the algorithm can be repeated $O(\log n / P(n))$ times, for an overall running time of $T(n) \cdot \frac{\log n}{P(n)} = O(n^2 \log^3 n)$. This is an order of magnitude improvement over Karger's original algorithm.

Finding all min-cuts

Theorem: With high probability we can find all min cuts in the running time of $O(n^2 \ln^3 n)$.

Proof: Since we know that $P(n) = O\left(\frac{1}{\ln n}\right)$, therefore after running this algorithm $O(\ln^2 n)$ times The probability of missing a specific min-cut is

$$\Pr[\text{miss a specific min-cut}] = (1 - P(n))^{O(\ln^2 n)} \le \left(1 - \frac{c}{\ln n}\right)^{\frac{3\ln^2 n}{c}} \le e^{-3\ln n} = \frac{1}{n^3}.$$

And there are at most $\binom{n}{2}$ min-cuts, hence the probability of missing any min-cut is

$$\Pr[\text{miss any min-cut}] \le \binom{n}{2} \cdot \frac{1}{n^3} = O\left(\frac{1}{n}\right).$$

The probability of failures is considerably small when *n* is large enough.■

Improvement bound

To determine a min-cut, one has to touch every edge in the graph at least once, which is $O(n^2)$ time in a dense graph. The Karger–Stein's min-cut algorithm takes the running time of $O(n^2 \ln^{O(1)} n)$, which is very close to that.

References

1. ^ ***a b*** Karger, David (1993). "Proc. 4th Annual ACM-SIAM Symposium on Discrete Algorithms" (http://people.csail.mit.edu/karger/Papers/mincut.ps). `|chapter=` ignored (help)
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4. ^ Karger, David R.; Stein, Clifford (1996). "A new approach to the minimum cut problem" (http://www.columbia.edu/~cs2035/courses/ieor6614.S09/Contraction.pdf). *Journal of the ACM* **43** (4): 601. doi:10.1145/234533.234534 (https://dx.doi.org/10.1145%2F234533.234534).

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