Wilf–Zeilberger pair

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In mathematics, specifically combinatorics, a **Wilf–Zeilberger pair**, or **WZ pair**, is a pair of functions that can be used to certify certain combinatorial identities. WZ pairs are named after Herbert S. Wilf and Doron Zeilberger, and are instrumental in the evaluation of many sums involving binomial coefficients, factorials, and in general any hypergeometric series. A function's WZ counterpart may be used to find an equivalent and much simpler sum. Although finding WZ pairs by hand is impractical in most cases, Gosper's algorithm provides a sure method to find a function's WZ counterpart, and can be implemented in a symbolic manipulation program.

Contents

Definition

Two functions F and G form a WZ pair if and only if the following two conditions hold:

$$\begin{aligned}&F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k),\\&\lim_{M\to\pm\infty}G(n,M)=0.\end{aligned}$$

Together, these conditions ensure that

$$\sum_{k=-\infty}^{\infty}[F(n+1,k)-F(n,k)]=0$$

because the function G telescopes:

$$\begin{aligned}\sum_{k=-\infty}^{\infty}[F(n+1,k)-F(n,k)]&=\lim_{M\to\infty}\sum_{k=-M}^{M}[F(n+1,k)-F(n,k)]\\&=\lim_{M\to\infty}\sum_{k=-M}^{M}[G(n,k+1)-G(n,k)]\\&=\lim_{M\to\infty}[G(n,M+1)-G(n,-M)]\\&=0-0\\&=0.\end{aligned}$$

Therefore,

$$\sum_{k=-\infty}^{\infty}F(n+1,k)=\sum_{k=-\infty}^{\infty}F(n,k),$$

that is

$$\sum_{k=-\infty}^{\infty}F(n,k)=\mathit{const}.$$

The constant does not depend on n. Its value can be found by substituting $n = n_0$ for a particular n_0.

If F and G form a WZ pair, then they satisfy the relation

$$G(n,k)=R(n,k)F(n,k-1),$$

where $R(n,k)$ is a rational function of n and k and is called the *WZ proof certificate*.

Example

A Wilf–Zeilberger pair can be used to verify the identity

$$\sum_{k=0}^{\infty}(-1)^k\binom{n}{k}\binom{2k}{k}4^{n-k}=\binom{2n}{n}.$$

Divide the identity by its right-hand side:

$$\sum_{k=0}^{\infty}\frac{(-1)^k\binom{n}{k}\binom{2k}{k}4^{n-k}}{\binom{2n}{n}}=1.$$

Use the proof certificate

$$R(n,k)=\frac{2k-1}{2n+1}$$

to verify that the left-hand side does not depend on n, where

$$\begin{aligned}F(n,k)&=\frac{(-1)^k\binom{n}{k}\binom{2k}{k}4^{n-k}}{\binom{2n}{n}},\\G(n,k)&=R(n,k)F(n,k-1).\end{aligned}$$

Now F and G form a Wilf–Zeilberger pair.

To prove that the constant in the right-hand side of the identity is 1, substitute $n = 0$, for instance.

References

- Marko Petkovsek, Herbert Wilf and Doron Zeilberger (1996). *A=B*. AK Peters. ISBN 1-56881-063-6.
- Tefera, Akalu (2010), "What Is . . . a Wilf-Zeilberger Pair?" (PDF), *AMS Notices*, **57** (04): 508–509.

External links

- Gosper's algorithm (http://www.pnas.org/cgi/reprint/75/1/40.pdf) gives a method for generating WZ pairs when they exist.
- Generatingfunctionology (http://www.math.upenn.edu/~wilf/gfology2.pdf) provides details on the WZ method of identity certification.