Wilf's Snake-Oil Method

We begin with an example.

Example 1. Find a closed form (if one exists) of the sum below.

$$\sum_{k\geq 0}\binom{k}{n-k} \tag{1}$$

Notice that n is the free variable. So let $a_n = \sum_{k\geq 0}\binom{k}{n-k}$ and let $A(x) \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0}$. Then

$$\begin{aligned}
A(x) = \sum_n a_n x^n &= \sum_n \sum_{k\geq 0}\binom{k}{n-k}x^n \\
&= \sum_{k\geq 0}\sum_n \binom{k}{n-k}x^n \\
&= \sum_{k\geq 0} x^k \sum_n \binom{k}{n-k}x^{n-k} \\
&= \sum_{k\geq 0} x^k \sum_r \binom{k}{r}x^r \\
&= \sum_{k\geq 0} x^k (1+x)^k \\
&= \sum_{k\geq 0} (x+x^2)^k
\end{aligned}$$

So we have a geometric series with common ratio $x + x^2$. Thus

$$A(x) = \frac{1}{1-x-x^2}$$

It follows that

$$a_n = \sum_{k\geq 0}\binom{k}{n-k} = f_n$$

where the f_n's are the Fibonacci numbers.

The Snake Oil Method for Managing Combinatorial Sums

a. Identify the free variable in the sum and name the sum. For example, $a_n = \sum_{k\geq 0} \binom{k}{n-k}$ in the introductory example.

b. Let $A(x) \overset{\text{ogf}}{\longleftrightarrow} \{a_n\}_{n\geq 0}$. Then $a_n = [x^n]A(x)$.

c. Now $A(x)$ is a double sum over n. Interchange the order of summation so that the inner sum has a simple closed form. It will be useful to have a catalogue of series whose closed forms are known (see 2.5 of the text). We list a few of the more common forms below.

d. Finally, try to identify the coefficients of the result.

A Few Useful Power Series

$$\sum_k \binom{n}{k} x^k = (1+x)^n \tag{2}$$

$$\sum_n \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}} \tag{3}$$

$$\sum_n \frac{1}{n+1}\binom{2n}{n} x^n = \frac{1-\sqrt{1-4x}}{2x} \tag{4}$$

We will also avoid specifying limits on indices whenever possible. For example, if n is a positive integer, we will write

$$2^n = \sum_k \binom{n}{k}$$

since the summand vanishes unless $0 \leq k \leq n$. This allows us to carry out the following manipulation without obsessing over the ranges of our variables of summation. For example,

$$\begin{aligned}\sum_k \binom{n}{r+k} x^k &= x^{-r} \sum_k \binom{n}{r+k} x^{r+k} \\ &= x^{-r} \sum_s \binom{n}{s} x^s \\ &= x^{-r}(1+x)^n\end{aligned}$$

We demonstrate the technique with a few more examples.

Example 2. Evaluate the sum

$$\sum_k \binom{n+k}{2k} 2^{n-k}, \quad n \geq 0$$

Let $\alpha \in \mathbb{R}$ and $a_n = a_n(\alpha) = \sum_k \binom{n+k}{2k} \alpha^{n-k}$. Also, let $A(x) \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0}$. Then

$$\begin{aligned}
A(x) = \sum_n a_n x^n &= \sum_n \sum_k \binom{n+k}{2k} \alpha^{n-k} x^n \\
&= \sum_k \sum_n \binom{n+k}{2k} \alpha^{n-k} x^n \\
&= \sum_k \alpha^{-k} \sum_n \binom{n+k}{2k} \alpha^n x^n \\
&= \sum_k \alpha^{-2k} x^{-k} \sum_n \binom{n+k}{2k} (\alpha x)^{n+k} \\
&= \sum_k \alpha^{-2k} x^{-k} \frac{(\alpha x)^{2k}}{(1-\alpha x)^{2k+1}}, \qquad \text{(by (3))} \\
&= \sum_k \frac{x^k}{(1-\alpha x)^{2k+1}} \\
&= \frac{1}{1-\alpha x} \sum_k \left(\frac{x}{(1-\alpha x)^2} \right)^k \\
&= \frac{1}{1-\alpha x} \frac{1}{1 - \dfrac{x}{(1-\alpha x)^2}} \\
&= \frac{1}{1-\alpha x} \frac{(1-\alpha x)^2}{(1-2x)^2 - x} \\
&= \frac{1-\alpha x}{1-(1+2\alpha)x + \alpha^2 x^2}
\end{aligned}$$

Now let $\alpha = 2$. Then

$$\begin{aligned}
A(x) &= \frac{1-2x}{1-5x+4x^2} \\
&= \frac{1/3}{1-x} + \frac{2/3}{1-4x}
\end{aligned}$$

It follows that

$$a_n(2) = \frac{1 + 2 \cdot 4^n}{3}, \quad n \geq 0$$

The result in Example 2 depended on our ability to identify the sum $\sum_n \binom{n+k}{2k}(\alpha x)^{n+k}$. What about something like

$$\sum_k \binom{n}{k}\binom{2n}{n-k}$$

Since the free variable n appears in both binomial coefficients, it's difficult to see how changing the order of summation might help. Fortunately, there is another way.

Example 3. Evaluate the

$$\sum_k \binom{n}{k}\binom{2n}{n-k} \tag{5}$$

Here we make the odd choice to replace all but one appearance of n with new free variables. So let

$$a_n = \sum_k \binom{n}{k}\binom{m}{s-k} \tag{6}$$

Then

$$\begin{aligned} A(x) &= \sum_n \sum_k \binom{n}{k}\binom{m}{s-k} x^n \\ &= \sum_k \binom{m}{s-k} \sum_n \binom{n}{k} x^n \\ &= \sum_k \binom{m}{s-k} \frac{x^k}{(1-x)^{k+1}} \end{aligned}$$

Unfortunately, this doesn't look terribly inviting. Let's try summing on one of the other free variables in (5). So let

$$b_m = \sum_k \binom{n}{k}\binom{m}{s-k}$$

and $B(x) \overset{\text{ogf}}{\longleftrightarrow} \{b_m\}_{m\geq 0}$. Then

$$\begin{aligned} B(x) &= \sum_m b_m x^m \\ &= \sum_m \sum_k \binom{n}{k}\binom{m}{s-k} x^m \\ &= \sum_k \binom{n}{k} \sum_n \binom{m}{s-k} x^m \\ &= \sum_k \binom{n}{k} \frac{x^{s-k}}{(1-x)^{s-k+1}} \\ &= \frac{x^s}{(1-x)^{s+1}} \sum_k \binom{n}{k} u^k, \qquad u = \frac{1-x}{x} \end{aligned}$$

Now that looks better. Did you see what happened? Summing on m allowed us to move the troublesome index $s-k$ outside of the binomial coefficient, where it was easier to deal with. Continuing, we have

$$\begin{aligned} B(x) &= \frac{u^{-s}}{1-x}\sum_k \binom{n}{k} u^k \\ &= \frac{u^{-s}}{1-x}(1+u)^n \end{aligned}$$

Substituting $s=n$ yields

$$\begin{aligned} &= \frac{1}{1-x}\left(\frac{1}{u}+1\right)^n \\ &= \frac{1}{1-x}\,\frac{1}{(1-x)^n} \end{aligned}$$

Finally,

$$\begin{aligned} \sum_k \binom{n}{k}\binom{2n}{n-k} = a_{2n} &= [x^{2n}]\frac{1}{(1-x)^{n+1}} \\ &= [x^{2n}]\frac{1}{x^n}\,\frac{x^n}{(1-x)^{n+1}} \\ &= [x^{2n+n}]\sum_r \binom{r}{n} x^r \\ &= \binom{3n}{n} \end{aligned}$$

You can compare a few values of the last expression with (5) by visiting the URLs: `https://tinyurl.com/y97goulm` and `https://tinyurl.com/ydf2gyyh`.