Spectral graph theory makes it possible to consider the properties of a graph using the properties of the matrices associated with it (adjacency, Laplacian, incidence). The point is that the adjacency matrix and the Laplacian of an undirected graph are symmetric, that is, they have complete sets of real eigenvalues and eigenvectors.

The prerequisites are rather mild and include basic linear algebra and combinatorics.

Introduction. We introduce our main heroes, id est Laplacian and adjacency matrices of a graph. Also, we introduce Courant–Fischer inequality which is the main tool for estimating eigenvalues.

Examples and strongly regular graphs. We consider a lot of explicit examples of graphs. Then we show how to apply basic spectral graph theory to the class of strongly regular graphs. Recall that a graph G is strongly regular every vertex has degree k, and also there are integers λ and μ such that (i) every two adjacent vertices have λ common neighbours; (ii) every two non-adjacent vertices have μ common neighbours.

Cauchy interlacing theorem. The application of the mentioned classical theorem to a proper matrix led Huang (in 2019!) to a very short proof of the famous and long-standing sensitivity conjecture. Namely, he shows that every $2^{n-1}+1$-vertex induced subgraph of the n-dimensional cube graph has a maximum degree at least $\sqrt{n}$.

Maximal clique of a graph. Motzkin–Straus theorem determines the maximal clique of a graph in terms of the maximum of the corresponding quadratic form over the unit simplex. It gives an algebraic proof of Turán theorem on the maximal number of edges in a graph without K_{r+1}.

The independence number of a graph. We start from the celebrated Lovász bound on the independence number which led him to a very short computation of Shannon capacity of the cycle C_5. Then we discuss several combinatorial applications (Paley graph, Kneser graph, eventown graph).

The chromatic number of a graph. Wilf provided a Brooks-type spectral theorem. Then we discuss Hoffman's lower bound on the chromatic number of a graph.

Eigenvalues and Graph Structure. Cuts, partitions and Cheeger's inequality.

Spanning trees and matchings. Kirchhoff's matrix tree theorem shows that the number of spanning trees is the determinant of a submatrix of the Laplacian matrix of the graph. Also we discuss several extensions of this theorem.

Google page rank. We consider a random walk on "pages" of a "network" and show how to construct a ranking based on the random walk. Here we use the Perron–Frobenius theorem.

Expanders. A sequence of graphs is called *expanders* if it has some properties of random graphs. There are a lot of equivalent definitions (in particular a spectral one), some of them seem stronger than others. By the definition a sequence of random graphs is expander with high probability. We consider several explicit constructions which are extremely useful in applications.

Applications of expanders. We apply expanders to coding theory and to constructions of random generators.

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