Euler number

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In mathematics, the **Euler numbers** are a sequence E_n of integers (sequence A122045 in the OEIS) defined by the Taylor series expansion

$$\frac{1}{\cosh t} = \frac{2}{e^{t} + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \cdot t^n,$$

where $\cosh t$ is the hyperbolic cosine. The Euler numbers appear as a special value of the Euler polynomials.

The odd-indexed Euler numbers are all zero. The even-indexed ones (sequence A028296 in the OEIS) have alternating signs. Some values are:

$$\begin{aligned}
E_0 &= 1\\
E_2 &= -1\\
E_4 &= 5\\
E_6 &= -61\\
E_8 &= 1\,385\\
E_{10} &= -50\,521\\
E_{12} &= 2\,702\,765\\
E_{14} &= -199\,360\,981\\
E_{16} &= 19\,391\,512\,145\\
E_{18} &= -2\,404\,879\,675\,441
\end{aligned}$$

Some authors re-index the sequence in order to omit the odd-numbered Euler numbers with value zero, or change all signs to positive. This article adheres to the convention adopted above.

The Euler numbers appear in the Taylor series expansions of the secant and hyperbolic secant functions. The latter is the function in the definition. They also occur in combinatorics, specifically when counting the number of alternating permutations of a set with an even number of elements.

Contents

Explicit formulas

As an iterated sum

An explicit formula for Euler numbers is:[1]

$$E_{2n} = i \sum_{k=1}^{2n+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j (k-2j)^{2n+1}}{2^k i^k k}$$

where i denotes the imaginary unit with $i^2 = -1$.

As a sum over partitions

The Euler number E_{2n} can be expressed as a sum over the even partitions of $2n$,[2]

$$E_{2n} = (2n)! \sum_{0 \le k_1, \ldots, k_n \le n} \binom{K}{k_1, \ldots, k_n} \delta_{n, \sum m k_m} \left(-\frac{1}{2!}\right)^{k_1} \left(-\frac{1}{4!}\right)^{k_2} \cdots \left(-\frac{1}{(2n)!}\right)^{k_n},$$

as well as a sum over the odd partitions of $2n-1$,[3]

$$E_{2n} = (-1)^{n-1} (2n-1)! \sum_{0 \le k_1, \ldots, k_n \le 2n-1} \binom{K}{k_1, \ldots, k_n} \delta_{2n-1, \sum (2m-1) k_m} \left(-\frac{1}{1!}\right)^{k_1} \left(\frac{1}{3!}\right)^{k_2} \cdots \left(\frac{(-1)^n}{(2n-1)!}\right)^{k_n},$$

where in both cases $K = k_1 + \cdots + k_n$ and

$$\binom{K}{k_1, \ldots, k_n} \equiv \frac{K!}{k_1! \cdots k_n!}$$

is a multinomial coefficient. The Kronecker deltas in the above formulas restrict the sums over the ks to $2k_1 + 4k_2 + \cdots + 2nk_n = 2n$ and to $k_1 + 3k_2 + \cdots + (2n-1)k_n = 2n-1$, respectively.

As an example,

$$\begin{aligned}
E_{10} &= 10! \left(-\frac{1}{10!} + \frac{2}{2!\,8!} + \frac{2}{4!\,6!} - \frac{3}{2!^2\,6!} - \frac{3}{2!\,4!^2} + \frac{4}{2!^3\,4!} - \frac{1}{2!^5}\right)\\
&= 9! \left(-\frac{1}{9!} + \frac{3}{1!^2\,7!} + \frac{6}{1!\,3!\,5!} + \frac{1}{3!^3} - \frac{5}{1!^4\,5!} - \frac{10}{1!^3\,3!^2} + \frac{7}{1!^6\,3!} - \frac{1}{1!^9}\right)\\
&= -50\,521.
\end{aligned}$$

As a determinant

E_{2n} is also given by the determinant

$$E_{2n} = (-1)^n (2n)! \begin{vmatrix}
\frac{1}{2!} & 1 & & & \\
\frac{1}{4!} & \frac{1}{2!} & 1 & & \\
\vdots & & \ddots & \ddots & \\
\frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & & \frac{1}{2!} & 1\\
\frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!}
\end{vmatrix}.$$

Asymptotic approximation

The Euler numbers grow quite rapidly for large indices as they have the following lower bound

$$|E_{2n}| > 8 \sqrt{\frac{n}{\pi}} \left(\frac{4n}{\pi e}\right)^{2n}.$$

Euler zigzag numbers

The Taylor series of $\sec x + \tan x$ is

$$\sum_{n=0}^{\infty} \frac{A_n}{n!} x^n,$$

where A_n is the Euler zigzag numbers, beginning with

1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, 2702765, 22368256, 199360981, 1903757312, 19391512145, 209865342976, 2404879675441, 29088885112832, ... (sequence A000111 in the OEIS)

For all even n,

$$A_n = (-1)^{\frac{n}{2}} E_n,$$

where E_n is the Euler number; and for all odd n,

$$A_n = (-1)^{\frac{n-1}{2}} \frac{2^{n+1}\left(2^{n+1}-1\right) B_{n+1}}{n+1},$$

where B_n is the Bernoulli number.

For every n,

$$\frac{A_{n-1}}{(n-1)!} \sin\left(\frac{n\pi}{2}\right) + \sum_{m=0}^{n-1} \frac{A_m}{m!(n-m-1)!} \sin\left(\frac{m\pi}{2}\right) = \frac{1}{(n-1)!}.$$

Generalized Euler numbers

Generalizations of Euler numbers include poly-Euler numbers and multi-poly-Euler numbers, which play an important role in multiple zeta functions.[4]

See also

- Bell number
- Bernoulli number
- Euler–Mascheroni constant

References

1. Ross Tang, "An Explicit Formula for the Euler zigzag numbers (Up/down numbers) from power series" (http://www.voofie.com/content/117/an-explicit-formula-for-the-euler-zigzag-numbers-updown-numbers-from-power-series/) Archived (https://web.archive.org/web/20120511083735/http://www.voofie.com/content/117/an-explicit-formula-for-the-euler-zigzag-numbers-updown-numbers-from-power-series/) 2012-05-11 at the Wayback Machine.
2. Vella, David C. (2008). "Explicit Formulas for Bernoulli and Euler Numbers". *Integers*. **8** (1): A1.
3. Malenfant, J. "Finite, Closed-form Expressions for the Partition Function and for Euler, Bernoulli, and Stirling Numbers". arXiv:1103.1585.
4. Jolany, Hassan; Corcino, Roberto B.; Komatsu, Takao (Oct 2015). "More properties on multi-poly-Euler polynomials". *Boletín de la Sociedad Matemática Mexicana*. **21** (2): 149–162.

External links

- Hazewinkel, Michiel, ed. (2001), "Euler numbers", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- Weisstein, Eric W. "Euler number". *MathWorld*.