Necklace (combinatorics)

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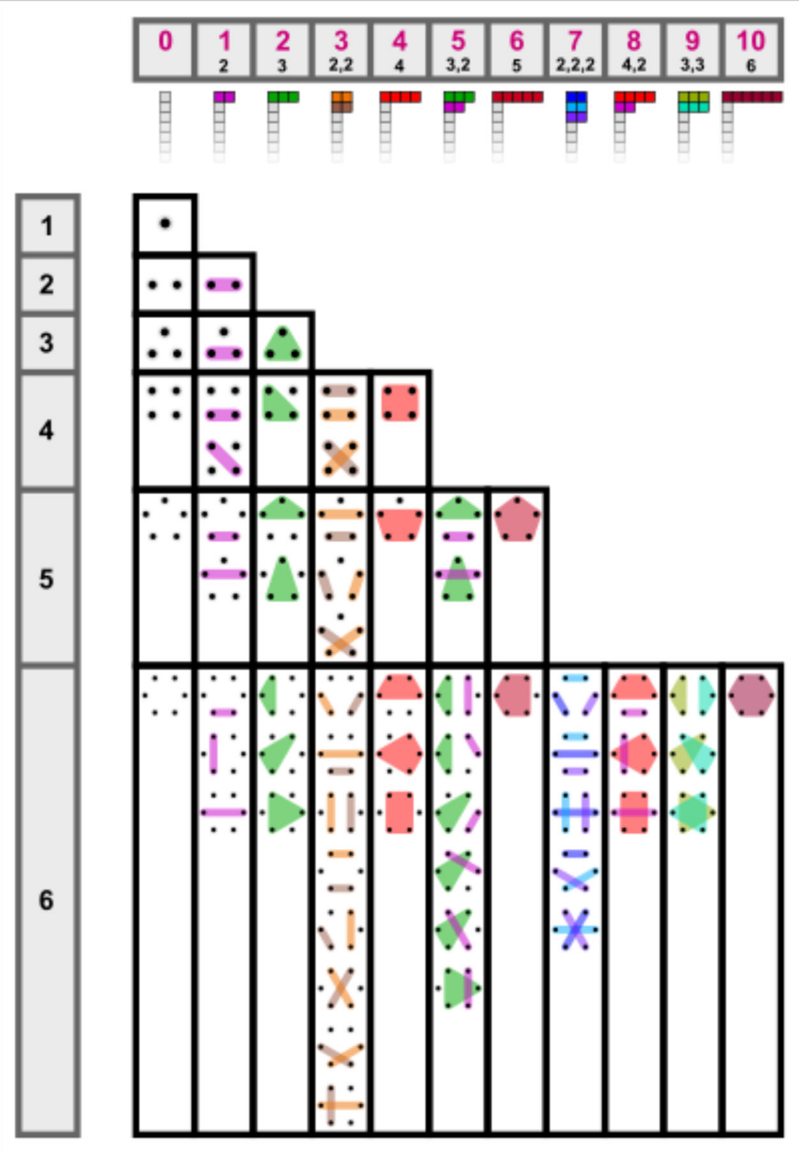
In combinatorics, a k-ary **necklace** of length n is an equivalence class of n-character strings over an alphabet of size k, taking all rotations as equivalent. It represents a structure with n circularly connected beads of up to k different colors.

A k-ary **bracelet**, also referred to as a **turnover** (or **free**) **necklace**, is a necklace such that strings may also be equivalent under reflection. That is, given two strings, if each is the reverse of the other then they belong to the same equivalence class. For this reason, a necklace might also be called a **fixed necklace** to distinguish it from a turnover necklace.

Technically, one may classify a necklace as an orbit of the action of the cyclic group on n-character strings, and a bracelet as an orbit of the dihedral group's action. This enables application of Pólya enumeration theorem for enumeration of necklaces and bracelets.



Possible patterns of bracelets of length n corresponding to the k-th integer partition (set partitions up to rotation and reflection)

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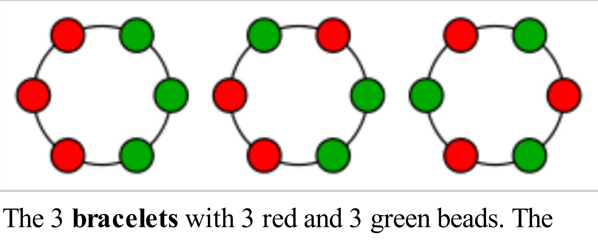
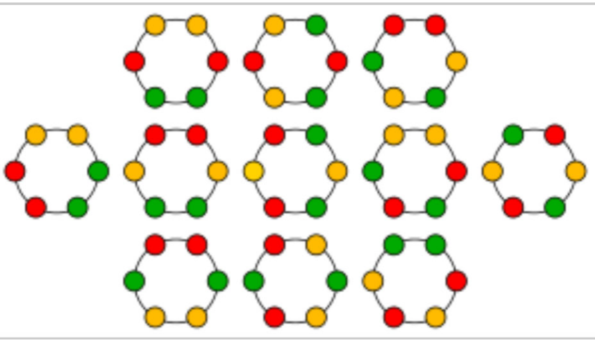
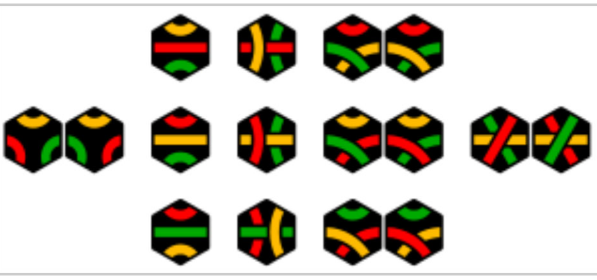


The 3 **bracelets** with 3 red and 3 green beads. The one in the middle is chiral, so there are 4 **necklaces**. Compare box(6,9) in the triangle.



The 11 **bracelets** with 2 red, 2 yellow and 2 green beads. The leftmost one and the four rightmost ones are chiral, so there are 16 **necklaces**. Compare box(6,7) in the triangle.



16 Tantrix tiles, corresponding to the 16 **necklaces** with 2 red, 2 yellow and 2 green beads.

Equivalence classes

Number of necklaces

There are

$$N_k(n) = \frac{1}{n}\sum_{d\mid n}\varphi(d)k^{\frac{n}{d}}$$

different k-ary necklaces of length n, where φ is Euler's totient function.[1]

Number of bracelets

There are

$$B_k(n) = \begin{cases} \frac{1}{2}N_k(n) + \frac{1}{4}(k+1)k^{\frac{n}{2}} & \text{if } n \text{ is even} \\ \frac{1}{2}N_k(n) + \frac{1}{2}k^{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

different k-ary bracelets of length n, where $N_k(n)$ is the number of k-ary necklaces of length n.

Examples

Necklace example

If there are n beads, all distinct, on a necklace joined at the ends, then the number of distinct orderings on the necklace, after allowing for rotations, is $\frac{n!}{n}$, for $n > 0$. This may also be expressed as $(n-1)!$. This number is less than the general case, which lacks the requirement that each bead must be distinct.

An intuitive justification for this can be given. If there is a line of n distinct objects ("beads"), the number of combinations would be $n!$. If the ends are joined together, the number of combinations are divided by n, as it is possible to rotate the string of n beads into n positions.

Bracelet example

If there are n beads, all distinct, on a bracelet joined at the ends, then the number of distinct orderings on the bracelet, after allowing for rotations and reflection, is $\frac{n!}{2n}$, for $n > 2$. Note that this number is less than the general case of $B_n(n)$, which lacks the requirement that each bead must be distinct.

To explain this, one may begin with the count for a necklace. This number can be further divided by 2, because it is also possible to flip the bracelet over.

Aperiodic necklaces

An **aperiodic necklace** of length n is an equivalence class of size n, i.e., no two distinct rotations of a necklace from such class are equal.

According to Moreau's necklace-counting function, there are

$$M_k(n) = \frac{1}{n}\sum_{d\mid n}\mu(d)k^{\frac{n}{d}}$$

different k-ary aperiodic necklaces of length n, where μ is the Möbius function.

Each aperiodic necklace contains a single Lyndon word so that Lyndon words form representatives of aperiodic necklaces.

Products of necklaces

The limit of the product of the numbers of fixed necklaces of length n composed of k types of beads:

$$\lim_{n\to\infty}\prod_{n=1}^{n} N_k(n) = \frac{k^n}{n!}1(1+X)\left(1+X+X^2\right)\cdots\left(1+X+X^2+\cdots+X^{n-1}\right)$$

where the coefficient of X^k in the expansion of the product

$$\prod_{m=1}^{n}\sum_{i=0}^{m-1}X^i = 1(1+X)\left(1+X+X^2\right)\cdots\left(1+X+X^2+\cdots+X^{n-1}\right)$$

presents the number of permutations of n with k inversions, expressed by a Mahonian number: A008302 (See Gaichenkov link)

See also

- Lyndon word
- Inversion (discrete mathematics)
- Necklace problem
- Necklace splitting problem
- Permutation
- Proofs of Fermat's little theorem#Proof by counting necklaces
- Forte number, a representation of binary bracelets of length 12 used in atonal music.

References

1. Weisstein, Eric W. "Necklace". *MathWorld*.

External links

- Weisstein, Eric W. "Necklace". *MathWorld*.
- Info on necklaces, Lyndon words, De Bruijn sequences (http://www.theory.csc.uvic.ca/~cos/inf/neck/NecklaceInfo.html)