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Note

A simple proof of Dixon's identity

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Abstract

We present another simple proof of Dixon's identity.
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Dixon [1] established the following famous identity

$$\sum_{k=-a}^{a} (-1)^k \binom{a+b}{a+k} \binom{b+c}{b+k} \binom{c+a}{c+k} = \frac{(a+b+c)!}{a!b!c!}, \tag{1}$$

where a, b, c are nonnegative integers (for short proofs, cf. [2,3]).

In this note, we give another simple proof of (1) in its polynomial version.

Theorem 1. *Let m, r be nonnegative integers, and x an indeterminate. Then*

$$\sum_{k=0}^{2r} (-1)^k \binom{m+2r}{m+k} \binom{x}{k} \binom{x+m}{m+2r-k} = (-1)^r \binom{x}{r} \binom{x+m+r}{m+r}. \tag{2}$$

Here and in what follows:

$$\binom{x}{r} = \frac{x(x-1)\cdots(x-r+1)}{r!}.$$

Proof. Denote the left-hand side of (2) by $P(x)$. We want to show that

$$P(x) = 0 \quad \text{for} -m-r \leqslant x < r.$$

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(i) $x = 0, 1, \ldots, r-1$, we have $0 \leqslant x < k$ or $0 \leqslant x < 2r-k$. Hence, $\binom{x}{k} = 0$ or $\binom{x+m}{m+2r-k} = 0$.
(ii) $x = -m, -m+1, \ldots, -1$, we have $0 \leqslant x+m < m \leqslant m+2r-k$. So, $\binom{x+m}{m+2r-k} = 0$.
(iii) $x = -m-r, -m-r+1, \ldots, -m-1$. Set $x = -p-1$, where $p = m, m+1, \ldots, m+r-1$. Then,

$$\begin{aligned} P(-p-1) &= \sum_{k=0}^{2r} (-1)^{m-k} \binom{m+2r}{m+k} \binom{p+k}{k} \binom{p+2r-k}{m+2r-k} \\ &= \sum_{k=-m}^{2r} (-1)^{m-k} \binom{m+2r}{m+k} \binom{p+k}{p} \binom{p+2r-k}{p-m} \\ &= 0. \end{aligned}$$

The last identity holds because $\binom{p+k}{p}\binom{p+2r-k}{p-m}$ is a polynomial in k of degree $2p-m < m+2r$, and we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^i = 0, \quad 0 \leqslant i < n,$$

which is well-known.

Moreover, $P(r)$ has only one nonzero term $(-1)^r\binom{m+2r}{m+r}$. Thus, $P(x)$ coincides with $(-1)^r\binom{x}{r}\binom{x+m+r}{m+r}$ at $m+2r+1$ values of x. Hence they must be identical. This completes the proof. □

Set $m+2r = a+b$, $x = b+c$, $x+m = c+a$ in Theorem 1. Then multiplying (2) by $(-1)^b$, and changing k to $b+k$, we obtain the form (1).

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