Типови системи

Трифон Трифонов

λ-смятане и теория на доказателствата, 2016/17 г.

9 май 2017 г.

λ-смятане с прости типове

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  $\rho, \sigma \quad ::= \quad \alpha \mid \rho \Rightarrow \sigma$

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  $(\lambda_x M)N \xrightarrow{\beta} M[x \mapsto N]$

  $(\lambda_x M x) \xrightarrow{\eta} M$, където $x \notin \mathrm{FV}(M)$

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- Изразителна сила: разширени полиноми
  $$p(x) \quad ::= \quad 0 \mid 1 \mid x \mid p_1(x) + p_2(x) \mid p_1(x).p_2(x) \mid p_1(p_2(x)) \mid \min(x,1) \mid 1 - \min(x,1)$$

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- Произведението е "синтактична захар"

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- Наредба на типовете

  $$\frac{\sigma_2 \equiv \sigma_1[\vec{\alpha} \mapsto \vec{\rho}]}{\forall_{\vec{\alpha}} \sigma_1 \sqsubseteq \forall_{\vec{\beta}} \sigma_2}\ \vec{\beta} \notin FV(\forall_{\vec{\alpha}} \sigma_1)$$

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Система на Hindley-Milner: типова изводимост

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma, x : \rho \vdash M : \sigma}{\Gamma \vdash \lambda_x M : \rho \Rightarrow \sigma}$$

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Programming Computable Functionals (PCF): редукции

- Редукции

$$
\begin{array}{llllll}
(\lambda_x s)t & \xrightarrow{\beta} & s[x \mapsto t], & \qquad \mathsf{P}\,0 & \xrightarrow{\beta} & 0, \\
\mathsf{Split}\,\langle s, t\rangle\, f & \xrightarrow{\beta} & f\; s\; t, & \qquad \mathsf{P}\,(\mathsf{S} n) & \xrightarrow{\beta} & n
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\mathsf{Y}\; t & \stackrel{\beta}{\to} & t(\mathsf{Y}\; t) & & &
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Система T на Gödel: редукции

- Редукции

$$
\begin{array}{lcllcl}
(\lambda_x s)t & \xrightarrow{\beta} & s[x \mapsto t], & \mathcal{R}_{\mathtt{nat}}\, 0\, s\, t & \xrightarrow{\beta} & s, \\
\mathsf{Split}\, \langle s, t \rangle\, f & \xrightarrow{\beta} & f\, s\, t, & \mathcal{R}_{\mathtt{nat}}\, (\mathsf{S} n)\, s\, t & \xrightarrow{\beta} & t\, n\, (\mathcal{R}_{\mathtt{nat}}\, n\, s\, t),
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$$

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\mathrm{Cases}\, \mathrm{tt}\, s\, t & \stackrel{\beta}{\to} & s, & & & \\
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