where B_j is the j^{th} *Bernoulli number*. The Bernolli numbers are defined with the recurrence

$$B_0 = 1$$
$$B_m = -\frac{1}{m}\sum_{j=0}^{m-1}\binom{m+1}{j}B_j, \text{ for } m \in \mathbb{N}^+$$

For details on the summation formula (4.19) and plenty of information on the Bernoulli numbers, see [GKP94, pp. 283–290]. Just keep in mind that Knuth *et al.* denote the sum by $S_k(n)$ and define it as

$$S_k(n) = 0^k + 1^k + 2^k + \ldots + (n-1)^k$$

For our purposes in this manual it is sufficient to know that

$$1^k + 2^k + \ldots + n^k = \Theta(n^{k+1}) \tag{4.20}$$

which fact follows easily from (4.19). In fact, (4.19) is a polynomial of degree $k+1$ of n because the $\binom{k+1}{j}$ factor and the Bernoulli numbers are just constants and clearly the highest degree of n is $k+1$. Strictly speaking, we have not proved here formally that (4.19) is a degree $k+1$ polynomial of n because we have not shown that the coefficient before n^{k+1} is not zero. But that is indeed the case—see for instance [GKP94, (6.98), pp. 288].

NB Be careful to avoid the error of thinking that

$$1^k + 2^k + \ldots + n^k$$

is a degree k polynomial of n and thus erroneosly concluding that its order of growth is $\Theta(n^k)$. It is *not* a polynomial of n because a polynomial has an *a priori* fixed number of terms, while the above sum has n terms where n is the variable.

Using (4.19), we can easily derive

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \tag{4.21}$$
$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{4.22}$$
$$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} \tag{4.23}$$
$$1^4 + 2^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \tag{4.24}$$

□

Problem 81. *Let* T *be a binary heap of height* h *vertices. Find the minimum and maixmum number of vertices in* T.

Solution:
The vertices of any binary tree are partitioned into *levels*, the vertices from level number i being the ones that are at distance i from the root. By definition, every level i in T, except possibly for level h, is complete in the sense it has all the 2^i vertices possible. The last

level (number h) can have anywhere between 1 and 2^h vertices inclusive. If n denotes the number of vertices in the heap, it is the case that

$$\underbrace{2^0 + 2^1 + 2^2 + \ldots + 2^{h-1}}_{\text{the number of vertices in the complete levels}} + 1 \leq n \leq \underbrace{2^0 + 2^1 + 2^2 + \ldots + 2^{h-1}}_{\text{the number of vertices in the complete levels}} + 2^h$$

Since $2^0 + 2^1 + 2^2 + \ldots + 2^{h-1} = \frac{2^h - 1}{2-1} = 2^h - 1$, it follows that

$$2^h - 1 + 1 \leq n \leq 2^h - 1 + 2^h$$

$$2^h \leq n \leq 2^{h+1} - 1 \tag{4.25}$$

□

Problem 82. *Let* T *be a binary heap with* n *vertices. Find the height* h *of* T*.*

Solution:

$$2^h \leq n \leq 2^{h+1} - 1 \quad \text{see Problem 81, (4.25)}$$

$$2^h \leq n < 2^{h+1}$$

$$h \leq \lg n < h + 1 \quad \text{take lg of both sides}$$

Clearly,

$$h = \lfloor \lg n \rfloor \tag{4.26}$$

□

Problem 83. *Let* T *be a binary heap with* n *vertices. Find the number of leaves and the number of internal vertices of* T*.*

Solution:
Let h be the height of T. We know (4.26) that $h = \lfloor \lg n \rfloor$. Let V' be the vertices of T at level h. Let T'' be obtained from T by deleting V' (see Figure 4.2). Clearly, T'' is a complete binary tree of height $h - 1 = \lfloor \lg n \rfloor - 1$. The number of its vertices is

$$2^{\lfloor \lg n \rfloor - 1 + 1} - 1 = 2^{\lfloor \lg n \rfloor} - 1 \tag{4.27}$$

It follows

$$|V'| = n - (2^{\lfloor \lg n \rfloor} - 1) = n + 1 - 2^{\lfloor \lg n \rfloor} \tag{4.28}$$

The vertices at level $h - 1$ are $2^{h-1} = 2^{\lfloor \lg n \rfloor - 1}$. Those vertices are partitioned into V'', the vertices that have no children, and V''', the vertices that have a child or two children (see Figure 4.2). So,

$$|V''| + |V'''| = 2^{\lfloor \lg n \rfloor - 1} \tag{4.29}$$

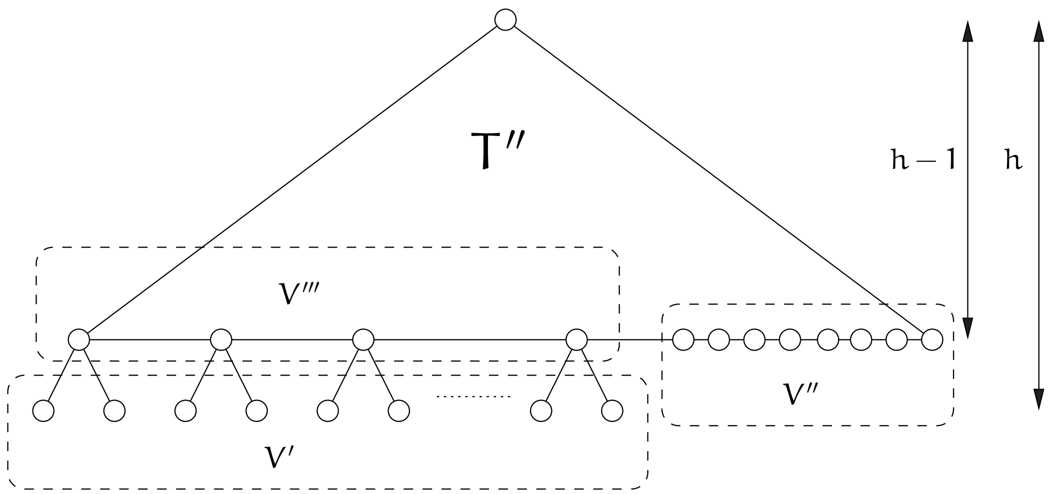

Figure 4.2: The heap in Problem 83.

Note that $|V'''| = \left\lceil \frac{|V'|}{2} \right\rceil$. Having in mind (4.28), it follows that

$$|V'''| = \left\lceil \frac{n+1-2^{\lfloor \lg n \rfloor}}{2} \right\rceil = \left\lceil \frac{n+1}{2} - \frac{2^{\lfloor \lg n \rfloor}}{2} \right\rceil = \left\lceil \frac{n+1}{2} - 2^{\lfloor \lg n \rfloor - 1} \right\rceil =$$

$$\left\lceil \frac{n+1}{2} \right\rceil - 2^{\lfloor \lg n \rfloor - 1} \qquad \text{since } 2^{\lfloor \lg n \rfloor - 1} \text{ is integer} \tag{4.30}$$

Use (4.29) and (4.30) to conclude that

$$\begin{aligned} |V''| &= 2^{\lfloor \lg n \rfloor - 1} - \left(\left\lceil \frac{n+1}{2} \right\rceil - 2^{\lfloor \lg n \rfloor - 1} \right) \\ &= 2^{\lfloor \lg n \rfloor - 1} - \left\lceil \frac{n+1}{2} \right\rceil + 2^{\lfloor \lg n \rfloor - 1} \\ &= 2.2^{\lfloor \lg n \rfloor - 1} - \left\lceil \frac{n+1}{2} \right\rceil \\ &= 2^{\lfloor \lg n \rfloor} - \left\lceil \frac{n+1}{2} \right\rceil \end{aligned} \tag{4.31}$$

It is obvious the leaves of T are $V' \cup V''$. Use (4.28) and (4.31) to conclude that

$$\begin{aligned} |V'| + |V''| = \; & n + 1 - 2^{\lfloor \lg n \rfloor} + 2^{\lfloor \lg n \rfloor} - \left\lceil \frac{n+1}{2} \right\rceil \\ &= n + 1 - \left\lceil \frac{n+1}{2} \right\rceil = n + 1 + \left\lfloor -\frac{n+1}{2} \right\rfloor \\ &= \left\lfloor n + 1 - \frac{n+1}{2} \right\rfloor \qquad \text{since } n+1 \text{ is integer} \\ &= \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil \end{aligned} \tag{4.32}$$

Then the internal vertices of T must be $\lfloor \frac{n}{2} \rfloor$ since $m = \lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil$ for any natural number m.

$\square$