3.3 Characterisations of planarity

In section 3.1 we proved that K_5 and $K_{3,3}$ are non-planar. These two graphs play a fundamental rôle in the classical characterisation of planarity due to Kuratowski and which is embodied in theorem 3.5. We use Kuratowski's theorem to establish two other descriptions of planarity which more precisely fit the requirements of this text. Before proceeding we need some definitions.

By $G_1 = (V_1, E_1)$ we denote a subgraph of $G = (V, E)$. A *piece* of G relative to G_1 is then:

either

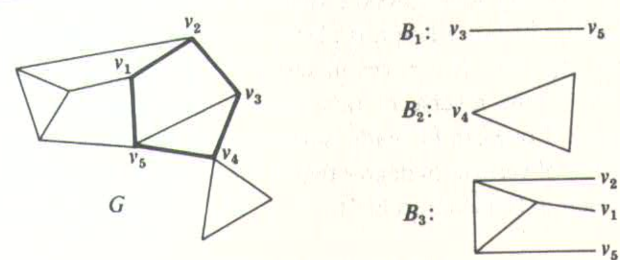
(*a*) an edge $(u, v) \in E$ where $(u, v) \notin E_1$ and $u, v \in V_1$,

or

(*b*) a connected component of $(G - G_1)$ plus any edges incident with this component.

In figure 3.7 the graph G has a subgraph G_1 which is a circuit $(v_1, v_2, v_3, v_4, v_5, v_1)$. B_1, B_2 and B_3 are the pieces of G relative to G_1. For any piece B, the vertices which B has in common with G_1 are called the *points of contact* of B. Thus in figure 3.7 B_1 has the points of contact v_3 and v_5, while B_3 has the points of contact v_1, v_2 and v_5. If a piece has two or more points of contact then it is called a *bridge*. Thus B_1 and B_3 are bridges but B_2 is not a bridge.

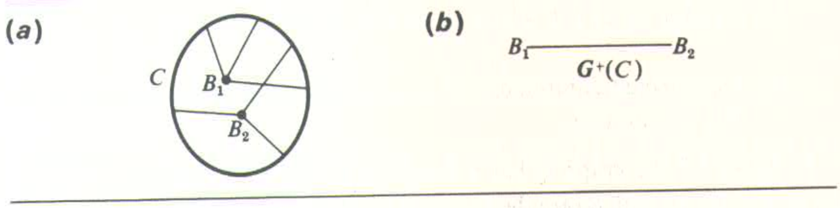
Fig. 3.7



Obviously a graph is planar if and only if each of its blocks is planar. Thus in questions of planarity we can always assume that we are dealing with blocks. Any piece of a block with respect to any proper subgraph is clearly a bridge.

Let C be any circuit which is a subgraph of G. $\tilde{C}$ then divides the plane into two faces, an *interior* face and an *exterior* face. For every pair of vertices of a given bridge of C, there is a path from one vertex to the other which does not use an edge of C. Of course, if G is planar, and if there exists a single bridge relative to C, then C is a boundary of some face because the bridge can belong to one and only one (namely, the other) face of C. Two bridges B_1 and B_2 are said to be *incompatible* ($B_1 \not\approx B_2$) if, when placed in the same face of the plane defined by C, at least two of their edges cross. See figure 3.8(*a*). To establish incompatibilities, each bridge is conveniently reduced to a single vertex connected to the points of contact with C.

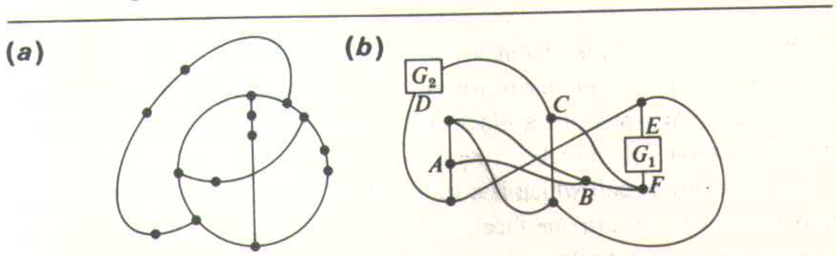
Fig. 3.8



An *auxiliary graph* $G^+(C)$ relative to a circuit C has a vertex-set consisting of a vertex for each bridge relative to C and an edge between any two such vertices B_i and B_j if and only if $B_i \not\approx B_j$. See, for example, figure 3.8(*b*). Suppose that $G^+(C)$ is a bipartite graph with bipartition $(B, \bar{B})$. Then the bridges in B may be embedded in one face of C and the bridges in $\bar{B}$ may be embedded in the other face. In this way no incompatible bridges occur in the same face.

Before presenting Kuratowski's theorem we need just one more definition. Whether or not a graph is planar is obviously unaffected either by dividing an edge into two edges in series by the insertion of a vertex of degree 2, or by the reverse of this process. Two graphs are said to be *homeomorphic* if one can be made isomorphic to the other by the addition or the deletion of vertices of degree two in this manner. Figure 3.9(*a*) shows a graph which is homeomorphic to $K_{3,3}$, while (*b*) shows a graph which

Fig. 3.9



contains a subgraph homeomorphic to $K_{3,3}$. In this second case the subgraph is obtained by deleting the edge (A, B), by replacing the connected subgraph G_1 by the path it contains from E to F and by similarly replacing the connected subgraph G_2 by a path from D to C.

Theorem [Kuratowski] 3.5. A graph is planar if and only if it has no subgraph homeomorphic to K_5 or to $K_{3,3}$.

Proof. In section 3.1 we proved that K_5 and $K_{3,3}$ are non-planar. It follows that any graph containing a subgraph homeomorphic to either cannot be planar.

It remains to be shown that a graph is planar if it does not contain a subgraph homeomorphic to K_5 or to $K_{3,3}$. We shall prove this by induction on the number of edges. It is clearly true for graphs with one or two edges. As the induction hypothesis we assume it to be true for all graphs with less than N edges. We now show that it is true for the graph G with N edges by demonstrating that the following statement leads to a contradiction: G is non-planar and does not contain a subgraph homeomorphic to K_5 or to $K_{3,3}$.

If G is non-planar, the following consequences apply:

(*a*) G must be connected. Otherwise G would consist of a number of components each with less than N edges, and each not having a subgraph homeomorphic to K_5 or $K_{3,3}$ (because G does not). By the induction hypothesis each component would be planar and hence so would G.

(*b*) G must not contain a point of articulation. If it did then G could be separated at this point of articulation, x. Each resulting component would be planar as in (*a*). For each component x could be mapped into the exterior face of a planar embedding according to theorem 3.2. The components could then clearly be rejoined at x without loss of planarity. Hence G would be planar.

(*c*) If any edge of G is removed, say (x, y), then the remaining graph G' contains a simple circuit passing through x and y. Notice that G' is connected because G contains no point of articulation. If no such simple circuit exists then every path from x to y would have to pass through a common vertex, say z. In other words, z would be an articulation point of G'. G' could then be separated at z into two components, G_1' (containing x) and G_2' (containing y). We add the edge (x, z) to G_1' so forming G_1'', and we add the edge (y, z) to G_2', so forming G_2''. Now neither G_1'' nor G_2'' could contain subgraphs homeomorphic to K_5 or to $K_{3,3}$ otherwise G would. This is because G contains a subgraph homeomorphic to G_1'', for example, where the path $(x, y, \ldots, z)$ in G takes the part of (x, z) in G_1''. By the induction hypothesis G_1'' and G_2'' would be planar. According to theorem 3.2 we could map (x, z) of G_1'' into the boundary of the exterior face of $\tilde{G}_1''$, similarly, we could take (y, z) of G_2'' to the exterior face of $\tilde{G}_2''$. Without loss of planarity, the two graphs G_1'' and G_2'' could then be joined at z and the edges (x, z) and (y, z) replaced by (x, y). This planar reconstruction of G thus yields a contradiction and so G' cannot contain an articulation point. G' is thus a block and so by theorem 2.10 contains a simple circuit passing through x and y.

Thus, summarising, $G' = G-(x, y)$ is connected and contains a simple circuit C passing through x and y. In fact C could be one of a number of such circuits. G' contains no subgraph homeomorphic to K_5 or to $K_{3,3}$, has one less edge than G and so, by the induction hypothesis, is planar. Let $\tilde{G}'$ be a planar embedding of G'. We then choose C to be the circuit passing through x and y which contains the largest number of faces of $\tilde{G}'$ in its interior. Any bridge of G' with respect to C is called an *interior* or an *exterior* bridge depending upon whether it lies in the interior or exterior of C for the embedding $\tilde{G}'$. For convenience we assign a direction to C which we take to be clockwise. If p and q are vertices on C, then $S[p, q]$ denotes the set of vertices from p to q (including p and q) on S going in a clockwise direction. $S]p, q[$ denotes $S[p, q]-\{p, q\}$. Note that no exterior bridge can have more than one point of contact in $S[x, y]$ or in $S[y, z]$. Otherwise C could be expanded to enclose at least one more face of $\tilde{G}'$.

G is constructed from the planar graph G' by adding the edge (x, y). Consider the requirements of exterior and interior bridges of $\tilde{G}'$ with respect to C in order that G be non-planar. There must exist at least one exterior bridge E and one interior bridge I. As fas as E is concerned there will be just two points of contact i and j with C such that:

$$i \in S]x, y[\quad \text{and} \quad j \in S]y, x[$$

I may have any number of points of contact with C. We certainly require that there are points of contact:

$$a \in S]x, y[\quad \text{and} \quad b \in S]y, x[$$

otherwise (x, y) may be added to the interior of C. We also require points of contact:

$$c \in S]i, j[\quad \text{and} \quad d \in S]j, i[$$

in order that $I \not\sim E$. In other words, I must be incompatible with E so that it cannot be taken into the exterior of C without loss of planarity.

Figure 3.10 schematically illustrates this. In this diagram a coincides with c and b coincides with d. There are however other possible configurations. Figure 3.11 illustrates all of those that are essentially different. For reasons of

Fig. 3.10

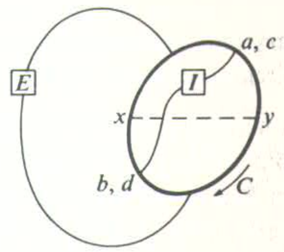
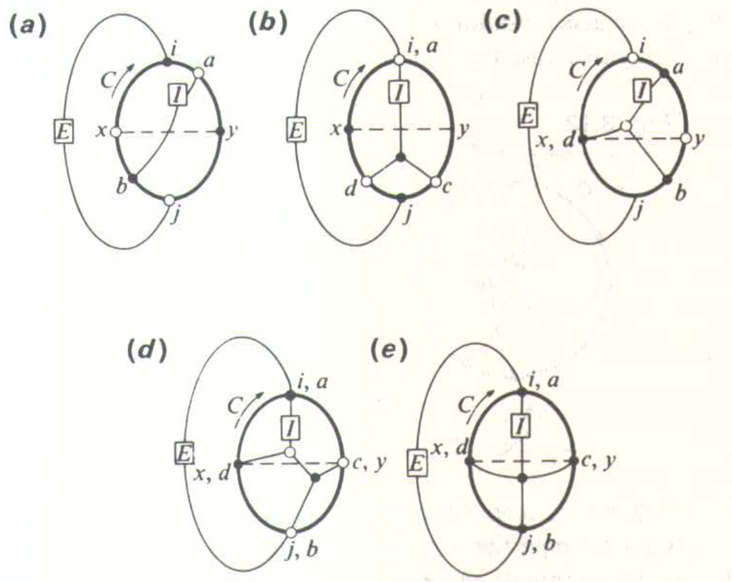


Fig. 3.11



clarity whenever any of a, b, c or d coincide, a single label is used. Notice that the configurations (d) and (e) differ only according to the internal paths in I linking a, b, c and d. Each of the configurations illustrated in (a), (b), (c) and (d) exhibit subgraphs which are homeomorphic to $K_{3,3}$. Open and closed circles are used to indicate the vertices of each partition.

The rather exceptional case indicated in (*e*) exhibits a subgraph homeomorphic to K_5. We have thus found the contradiction we were seeking and so the theorem is proved. ■

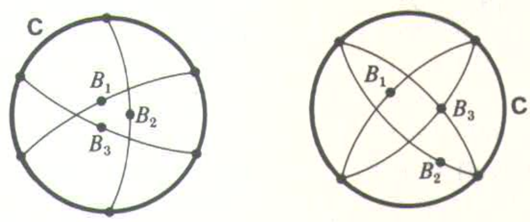
The following theorem provides a more appropriate insight into the nature of planarity as far as the planarity algorithm of section 3.4 is concerned.

Theorem 3.6. A necessary and sufficient condition for a graph G to be planar is that for every circuit C of G the auxiliary graph $G^+(C)$ is bipartite.

Proof. The condition is necessary because for any circuit C of a planar graph G, we can form a bipartition $(B, \bar{B})$ of the bridge vertices of G relative to C, such that bridges in B lie in one face of C for $\tilde{G}$, and the bridges of $\bar{B}$ lie in the other face. Clearly, $G^+(C)$ is bipartite because no edge of $G^+(C)$ connects two vertices in B or connects two vertices in $\bar{B}$.

That the condition is sufficient can be seen as follows. If G is not planar then according to Kuratowski's theorem G contains a subgraph homeomorphic to K_5 or to $K_{3,3}$. We suppose that G contains K_5 or $K_{3,3}$ as a subgraph, the generalisation to G containing proper homeomorphisms is obvious. In either case (see figure 3.12, in which the chosen circuits are

Fig. 3.12



$K_{3,3}$ K_5

indicated by heavily scored edges), we can choose C of the subgraph such that $G^+(C)$ is not bipartite. For $K_{3,3}$ there are three bridges B_1, B_2 and B_3, each of which is a single edge and any two of which are incompatible. In the case of K_5 there are again three bridges B_1, B_2 and B_3. B_1 and B_2 are single edges while B_3 is a vertex of K_5 plus its edges of attachment to C. Again any two of the bridges are incompatible. Thus for both K_5 and $K_{3,3}$, for the circuits chosen, $G^+(C) = K_3$ which is not bipartite. ■

The second characterisation of planarity of particular use in this text concerns *dual* graphs to which we devote the following section.

colouring areas of a map using the minimum number of colours such that no two adjacent regions are similarly coloured. It is now known that the famous 'four-colour' conjecture is true, namely, that four colours are sufficient. All we wish to note here is that the map colouring problem is precisely equivalent to the problem of colouring the vertices of the dual (of the graph corresponding to the map) such that no two adjacent vertices are similarly coloured. The dual graph provides a more convenient vehicle for reasoning about the problem.

3.4 A planarity testing algorithm

Before subjecting a particular graph to an algorithm which determines whether or not it is planar, some preprocessing may considerably simplify the task. In this connection we note the following points:

(*a*) If the graph is not connected then we subject each component to the test separately.

(*b*) If the graph is separable (that is, has one or more articulation points) then it is clearly planar if and only if each of its blocks is planar. We therefore disconnect the graph and subject each block separately to the test.

(*c*) Self-loops may obviously be removed without affecting planarity.

(*d*) Each vertex of degree 2 plus its incident edges can be replaced by a single edge. In other words, we construct the homeomorphic graph with the smallest number of vertices. This graph is clearly planar if and only if the original graph is planar.

(*e*) Parallel edges can clearly be removed without affecting planarity.

The last two simplifying steps ought to be applied repeatedly and alternately until neither can be applied further. Following these simplifications two elementary tests can be applied:

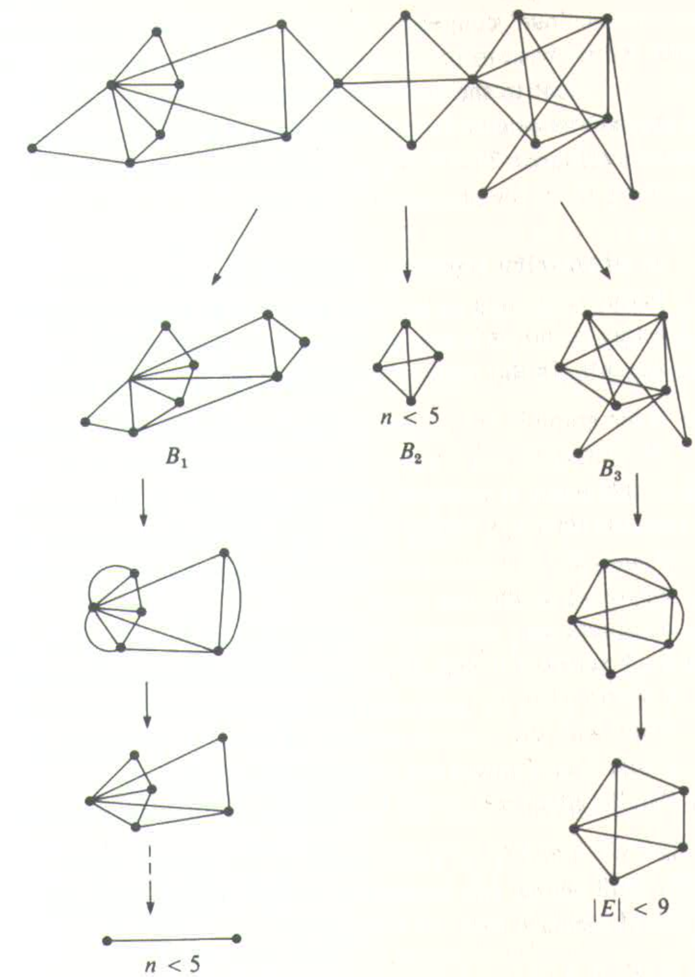
(*f*) If $|E| < 9$ or $n < 5$ then the graph must be planar.

(*g*) If $|E| > 3n-6$ then the graph, by corollary 3.1, must be non-planar.

If these two tests fail to resolve the question of planarity then the preprocessed graph is subjected to a more elaborate test. We pursue that shortly. First it is worth demonstrating what simplification can result from this preprocessing, particularly the repeated applications of (*d*) and (*e*). Figure 3.17 shows a graph with three blocks subjected to this processing which resolves that the graph is planar.

Many algorithms have been published which test for planarity. Planarity testing can be done in $O(n)$ time as Hopcroft & Tarjan[2] first showed.

Fig. 3.17

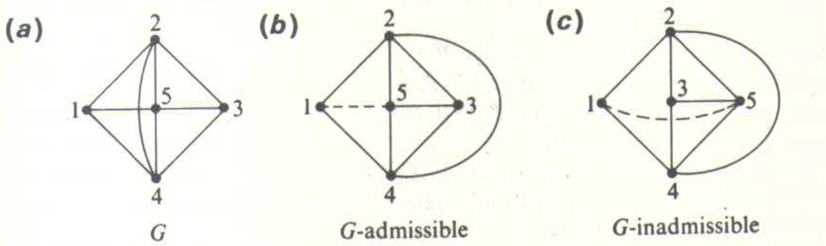


Lempel, Even & Cederbaum[3] published an algorithm which, through the work of Even & Tarjan[4] and Leuker & Booth[5] was also shown to be realisable in O(n)-time. These two algorithms require lengthy explanations and verification. We therefore describe a much simpler but nevertheless fairly efficient algorithm due to Demoucron, Malgrange & Pertuiset.[6] Of course, what is subjected to the algorithm, following any preprocessing, is a block. Before describing the algorithm we need one further definition.

Let $\tilde{H}$ be a planar embedding of the subgraph H of G. If there exists a planar embedding $\tilde{G}$, such that $\tilde{H} \subseteq \tilde{G}$, then $\tilde{H}$ is said to be *G-admissible*.

For example consider figure 3.18. In (*a*) a graph is shown while (*b*) and (*c*) show two different planar embeddings of the same subgraph $H = G-(1, 5)$. In (*b*) $\tilde{H}$ is G-admissible whilst (*c*) shows an embedding of H which is *not* G-admissible.

Fig. 3.18



Let B be any bridge of G relative to H. Now, B can be *drawn* in a face of $\tilde{H}$ if all the points of contact of B are in the boundary of f. By $F(B, H)$ we denote the set of faces of $\tilde{H}$ in which B is drawable.

The planarity testing algorithm is outlined in figure 3.19. The algorithm finds a sequence of graphs $G_1, G_2, \ldots$, such that $G_i \subset G_{i+1}$ and finds their planar embeddings $\tilde{G}_1, \tilde{G}_2, \ldots$. If G is planar then, as we shall see, each $\tilde{G}_i$ found by the algorithm is G-admissible and the algorithm terminates with a planar embedding of G, $\tilde{G}_{|E|-n+1}$. If G is non-planar then the algorithm stops with the discovery of some bridge B (with respect to the current G_i) for which $F(B, \tilde{G}_i) = \varnothing$. Obviously a necessary condition that $\tilde{G}_i$ is G-admissible is that for every bridge B relative to G_i, $F(B, \tilde{G}_i) \neq \varnothing$.

The first of the sequence of graphs found by the algorithm, G_1, is a circuit (lines 1–3). Since G is a block it must contain such a circuit. Clearly, G_1 will be planar. The boolean variable *EMBEDDABLE* (lines 5, 6, 10 and 12) has the value **true** so long as the algorithm has not detected a bridge B relative to the current $\tilde{G}_i$ for which $F(B, \tilde{G}_i) = \varnothing$. If it acquires the value **false** then the algorithm terminates (line 6) with the message 'G is non-planar' (line 11). The variable f is used to record the number of faces of the current $\tilde{G}_i$. It is initialised to the value 2 in line 4 and is incremented by one for each execution of the **while** body (lines 7–19). Each execution of the **while** body constructs a new $\tilde{G}_{i+1}$ from the current $\tilde{G}_i$. This is achieved as follows. Lines 7 and 8, respectively, find the set of bridges of G relative to G_i and for each such bridge B, the set $F(B, \tilde{G}_i)$. If there now exists a bridge B which can be drawn in *only one* face F of $\tilde{G}_i$ (i.e., $|F(B, G_i)| = 1$, line 13), then $\tilde{G}_{i+1}$ is constructed by drawing a path P_i between two points of contact of B in the face F. If no such bridge exists

Fig. 3.19. A planarity testing algorithm.

```
 1.  Find a circuit C of G
 2.  i ← 1
 3.  G₁ ← C, G̃₁ ← C
 4.  f ← 2
 5.  EMBEDDABLE ← true
 6.  while f ≠ |E| − n + 2 and EMBEDDABLE do
         begin
 7.      find each bridge B of G relative to Gᵢ
 8.      for each B find F(B, G̃ᵢ)
 9.      if for some B, F(B, G̃ᵢ) = ∅ then
             begin
10.          EMBEDDABLE ← false
11.          output the message 'G is non-planar'
             end
12.      if EMBEDDABLE then
             begin
13.          if for some B, |F(B, G̃ᵢ)| = 1 then F ← F(B, G̃ᵢ)
                 else let B be any bridge and F be any face such
                     that F ∈ F(B, G̃ᵢ)
14.          find a path Pᵢ ⊆ B connecting two points of contact
                 of B to Gᵢ
15.          Gᵢ₊₁ ← Gᵢ + Pᵢ
16.          Obtain a planar embedding G̃ᵢ₊₁ of Gᵢ₊₁ by drawing Pᵢ
                 in the face F of G̃ᵢ
17.          i ← i + 1
18.          f ← f + 1
19.          if f = |E| − n + 2 then output the message 'G is planar'
             end
         end.
```

then P_i is a path between two points of contact for *any* bridge. In either case, P_i divides some face F into two faces and f is incremented by one (line 18). Notice that if G is planar then $\tilde{G}$ will have, according to theorem 3.3, $(|E|-n+2)$ faces and this fact is used to terminate the algorithm (lines 6 and 19). In a more detailed encoding of the algorithm, each $\tilde{G}_i$ may be represented by its set of faces $\{F_i\}$. Here each F_i can be described by the ordered set of vertices which mark its boundary in, say, a clockwise direction about an axis passing through the face. In this sense of course, each axis ought to be viewed from the same side of the plane.

Of course, if the graph is planar, then the algorithm obtains a planar embedding, $G_{|E|-n+1}$, and this could be output in the form of a set of faces by a modification of the conditional statement 19.

Theorem 3.10. The algorithm of Demoucron *et al.* is valid.

Proof. We have to show that each term of the sequence $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_{|E|-n+1}$, if G is planar, is G-admissible. The proof is by induction. If G is planar then $\tilde{G}_1$ is clearly G-admissible. We assume that $\tilde{G}_i$ is G-admissible for $1 \leqslant i \leqslant k < |E|-n+1$. We now show that $\tilde{G}_{k+1}$ will be G-admissible. Let B and F be as defined in statement 13 of the algorithm. Let $\tilde{G}$ be a planar embedding of G where $\tilde{G}_k \subset \tilde{G}$. If $|F(B, \tilde{G}_k)| = 1$ then, clearly, $\tilde{G}_{k+1}$, as constructed by the algorithm satisfies $\tilde{G}_{k+1} \subseteq \tilde{G}$. We therefore suppose that $|F(B, \tilde{G}_k)| > 1$ and imagine that B is *not* drawn in F in $\tilde{G}$ but in some other face F'. Now G is a block so that every bridge of G with respect to G_k has at least two points of contact and can therefore be drawn in just two faces. Thus each bridge with points of contact on the boundary between the faces F and F' may be drawn individually in either F or in F'. Now there clearly exists another planar embedding of G in which each such bridge is drawn in F if it appears in F' in $\tilde{G}$ and is drawn in F' if it appears in F in $\tilde{G}$. The $\tilde{G}_{k+1}$ constructed by the algorithm is clearly G-admissible, since $\tilde{G}_{k+1}$ is contained in this new $\tilde{G}$. ∎

It is easy to see that the planarity testing algorithm can be implemented in polynomial time although it is less sophisticated than the linear-time algorithms mentioned earlier. We leave the details to the reader (exercise 3.14). However we note the following. The body of the *while* statement (lines 7–19) is executed at most $(|E|-n+1)$ times. In order to find each bridge B of $G = (V, E)$ relative to $G_i = (V_i, E_i)$ in line 7, we define $G' = (G - V_i)$, and then need to find:

(*a*) each $(u, v) \in E$ such that $(u, v) \notin E_i$, but $u \in V_i$ and $v \in V_i$,

and

(*b*) each component of G' and add to each component any edges that connect it to vertices in V_i.

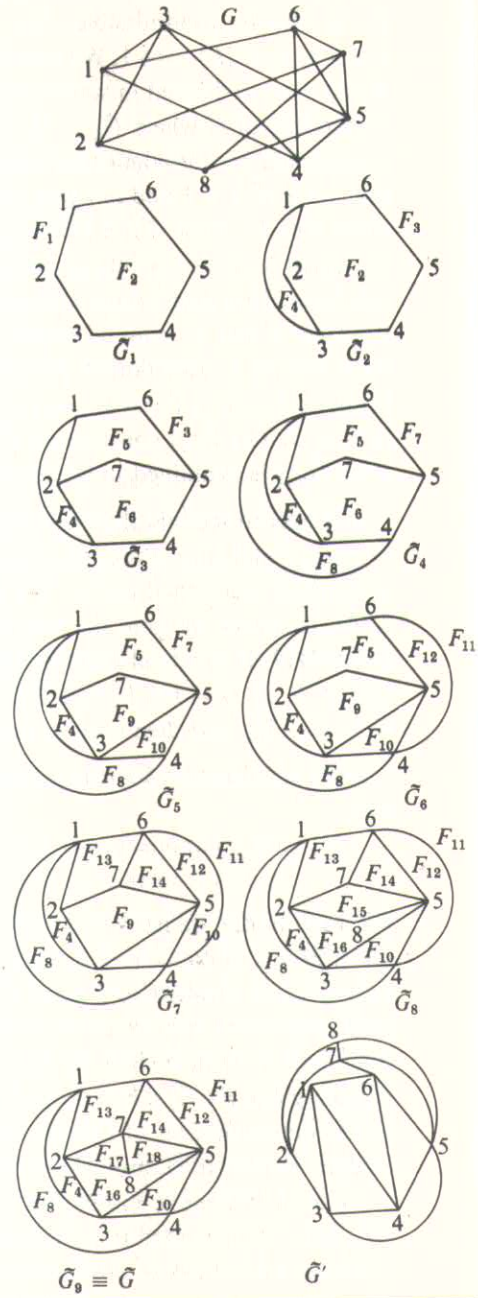
For each bridge we need to record its points of contact with G_i. If b is the set of points of contact of B, then in line 8, a face F is in $F(B, \tilde{G}_i)$ if and only if *every* element of b is in F. Here we presume that F denotes an (ordered) set of vertices as described earlier. If each face is described in this manner, then in line 16 $\tilde{G}_{i+1}$ is easily obtained from $\tilde{G}_i$ by simply replacing one $F \in \tilde{G}_{i+1}$ by two new faces in an obvious manner. Returning to the determination of bridges in line 7, notice that all but one of the bridges relative to G_i are bridges relative to $\tilde{G}_{i+1}$. This exceptional bridge is replaced by none or more other bridges. All other steps of the algorithm are easily implemented in an efficient manner.

Figure 3.20 shows an application of the algorithm to the graph G shown there. For each successive G_i, the diagram contains a tabulation of the set of bridges relative to G, the value of f, $F(B, \tilde{G}_i)$, B and F as defined in

Fig. 3.20. An application of the planarity testing algorithm.

$\tilde{G}_i$	f	Bridges	$F(B, \tilde{G}_i)$	B	F	P_i
$\tilde{G}_1$	2	B_1	$\{F_1, F_2\}$			
		B_2	$\{F_1, F_2\}$			
		B_3	$\{F_1, F_2\}$			
		B_4	$\{F_1, F_2\}$			
		B_5	$\{F_1, F_2\}$	B_1	F_1	(1, 3)
$\tilde{G}_2$	3	B_2	$\{F_2, F_3\}$			
		B_3	$\{F_2, F_3\}$			
		B_4	$\{F_2, F_3\}$			
		B_5	$\{F_2\}$	B_5	F_2	(2, 7, 5)
$\tilde{G}_3$	4	B_2	$\{F_3\}$			
		B_3	$\{F_3, F_6\}$			
		B_4	$\{F_3\}$			
		B_6	$\{F_5\}$			
		B_7	$\{F_5, F_6\}$	B_2	F_3	(1, 4)
$\tilde{G}_4$	5	B_3	$\{F_6\}$			
		B_4	$\{F_7\}$			
		B_6	$\{F_5\}$			
		B_7	$\{F_5, F_6\}$	B_3	F_6	(3, 5)
$\tilde{G}_5$	6	B_4	$\{F_7\}$			
		B_6	$\{F_5\}$			
		B_7	$\{F_5, F_9\}$	B_4	F_7	(4, 6)
$\tilde{G}_6$	7	B_6	$\{F_5\}$			
		B_7	$\{F_5, F_9\}$	B_6	F_5	(6, 7)
$\tilde{G}_7$	8	B_7	$\{F_9\}$	B_7	F_9	(2, 8, 5)
$\tilde{G}_8$	9	B_8	$\{F_{15}\}$	B_8	F_{15}	(7, 8)
$\tilde{G}_9$	10	$(\lvert E\rvert - n + 2) = 10 = f$: algorithm terminates				

Bridge definitions
$B_1 = [(1, 3)]$, $B_2 = [(1, 4)]$, $B_3 = [(3, 5)]$
$B_4 = [(4, 6)]$
$B_5 = [(7, 8), (7, 2), (7, 5), (7, 6), (8, 2), (8, 5)]$
$B_6 = [(6, 7)]$, $B_7 = [(8, 2), (8, 5), (8, 7)]$
$B_8 = [(7, 8)]$



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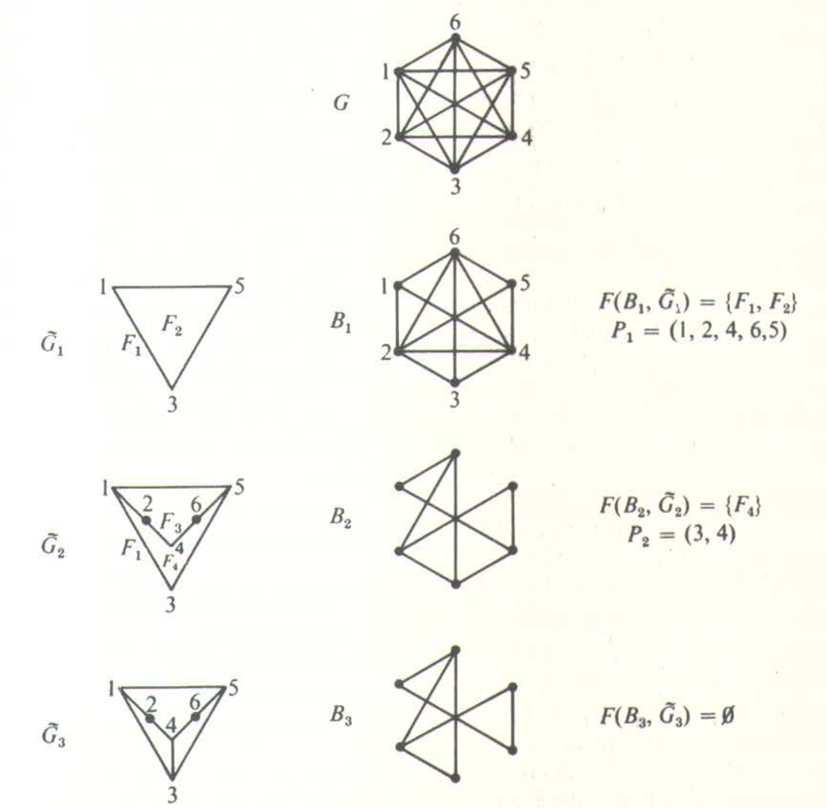
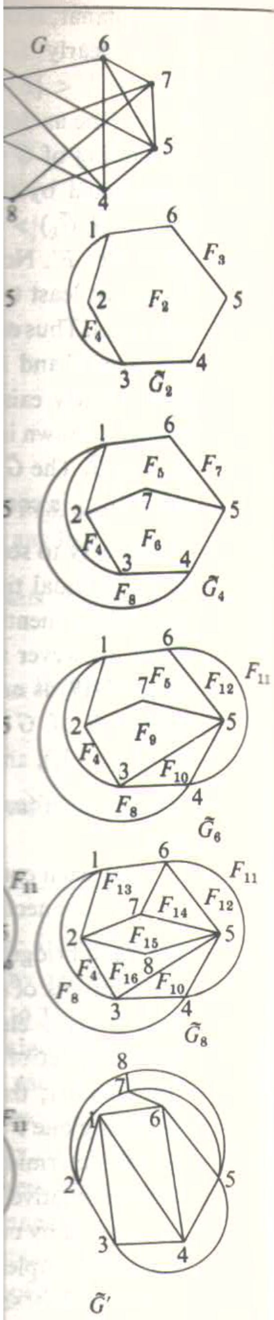
rather than in F_1.
Because G is plar
could all have b
example because
obtained from $\tilde{G}$
to become the ex
as defined in stat
can be obtained.

Fig. 3.2

statement 13 of the algorithm and P_i as defined in statement 14. There is a separate table defining each bridge by its edge-set. As can be seen, in this case the algorithm terminates when $f = (|E| - n + 2)$ with a planar embedding of G, $\tilde{G}_9$ and the message 'G is planar' would be output. The additional sketch labelled $\tilde{G}'$ represents a planar embedding of G which could have resulted if in going from $\tilde{G}_1$ to $\tilde{G}_2$ the path (1, 3) had been placed in F_2 rather than in F_1. This illustrates a point in the verification of theorem 3.10. Because G is planar, the bridges relative to $\tilde{G}_1$ that are finally placed in F_1 could all have been placed in F_2 and vice versa. This is rather a special example because $\tilde{G}'$ is not distinctly different from $\tilde{G}_9$. In fact, $\tilde{G}'$ can be obtained from $\tilde{G}_9$ merely by causing (see theorem 3.2) the face (2, 8, 5, 3) to become the exterior face. In general, however, given a choice of B and F as defined in statement 13 of the algorithm, distinctly different embeddings can be obtained.

Fig. 3.21. An application of the planarity testing algorithm.





Finally, Figure 3.21 shows an application of the algorithm to the non-planar graph K_6. For each $\tilde{G}_i$ there is one bridge denoted by B_i, $F(B_i, \tilde{G}_i)$ and P_i also indicated in each case. The algorithm terminates when $F(B_3, \tilde{G}_3) = \varnothing$ with the message 'G is non-planar'.

3.5 Summary and references

Euler's formula provides a simple basis for deriving many immediate results relating to planar graphs. Some of the problems that follow provide further illustration of this. We also provided the extension to non-planar surfaces in section 3.2. The treatment of non-planar surfaces was informal, being illustrative rather than rigorous. Results in this area are highly specific and not of much practical benefit. Chapter 2 of Beineke & Wilson[7] provides a good commentary and selection of results. Chapter 11 of Harary[8] is also worthy of a reference.

The main characterisations of planarity we described were those of Kuratowski[9] and of Whitney[10] who used the idea of combinatorial dual. Our proofs of the relevant theorems are not based upon the original papers but on simpler expositions. The proof of theorem 3.5 is largely based on one given by Berge,[11] whilst the proof of theorem 3.8 is based on Parsons'.[12] Another well-known characterisation of planarity not covered in the text is that due to McLane[13]: a graph is planar if and only if it has a circuit basis (see section 2.2.1), together with one additional circuit such that this collection of circuits contains each edge of the graph twice. Finally, theorem 3.6 is essentially taken from Demoucron *et al.*[6]

A survey of early planarity testing algorithms is provided by Shirey.[14] As was stated earlier, linear time algorithms have been described by Hopcroft & Tarjan[2] and by Lempel *et al.*[3] Both of these algorithms receive detailed description in Even.[15] Our validification in theorem 3.10 of the planarity testing algorithm of Demoucron *et al.*,[6] which is rather simpler than that to be found in the original text, was influenced by the presentation of Bondy & Murty in [16].

[1] Lipton, R. J. & Tarjan, R. E. 'Applications of a planar separator theorem', *SIAM J. Comput.*, **9** (3) (1980).
[2] Hopcroft, J. & Tarjan, R. E. 'Efficient planarity testing', *JACM*, **21** (4), 549–68 (1974).
[3] Lempel, A., Even, S. & Cederbaum, I. 'An algorithm for planarity testing of graphs', *Theory of Graphs, International Symposium*, Rome 1966, P. Rosenstiehl, ed., Gordon and Breach, NY, pp. 215–32 (1967).
[4] Even, S. & Tarjan, R. E. 'Computing on st-numbering', *Theory of Computer Science*, **2**, 339–44 (1976).
[5] Lueker, G. S. & Booth, K. S. 'Testing for the consecutive ones property, interval graphs and graph planarity using *PQ*-tree algorithms', *J. of Comp. and Sys. Sciences*, **13**, 335–79 (1976).

[6] Demoucron, G., Malgrange, Y. & Pertuiset, R. 'Graphes planaires: reconnaissance et construction de représentations planaires topologiques', *Rev. Francaise Recherche Opérationnelle*, **8**, 33–47 (1964).

[7] Beineke, L. W. & Wilson, R. J. (eds.), *Selected Topics in Graph Theory*, chapter 2 (White & Lowell). Academic Press (1978).

[8] Harary, F. *Graph Theory*. Addison-Wesley (1969).

[9] Kuratowski, G. 'Sur le problème des courbes gauches en topologie', *Fund. Math.*, **15**, 271–83 (1930).

[10] Whitney, H. (*a*) 'Non-separable and planar graphs', *Trans. Am. Maths. Soc.*, **34**, 339–62 (1932).
Whitney, H. (*b*) 'Planar graphs', *Fund. Maths.*, **21**, 73–84 (1933).
Whitney, H. (*c*) '2-isomorphic graphs', *Am. J. Maths*, **55**, 245–54 (1933).

[11] Berge, C. *Theory of Graphs and its Applications*. John Wiley and Sons, NY (1962).

[12] Parsons, T. D. 'On planar graphs', *Am. Maths. Monthly*, **78** (2), 176–8 (1971).

[13] McLane, S. 'A combinatorial condition for planar graphs', *Fund Maths.*, **28**, 22–32 (1937).

[14] Shirey, R. W. Implementation and analysis of efficient planarity testing algorithms. PhD dissertation. Computer Sciences. University of Wisconsin (1969).

[15] Even, S. *Graph Algorithms*. Computer Science Press (1979).

[16] Bondy, J. A. & Murty, U. S. R. *Graph Theory with Applications*. The Macmillan Press (1976).

[17] Fáry, I. 'On straight line representation of planar graphs', *Acta. Sci. Math. Szeged.*, 229–33 (1948).

[18] Filotti, I. S., Miller, G. L. & Reif, J. 'On determining the genus of a graph in $O(V^{O(g)})$ steps', *Proceedings of the eleventh annual ACM Symposium on the Theory of Computing*. Atlanta, Georgia (1979).

EXERCISES

3.1. Given an arbitrary simple planar graph with n vertices and $|E|$ edges, show that the maximum number of edges, M, that can be added to the graph, subject to it remaining planar is given by

$$M = 3n - |E| - 6$$

(Use Euler's formula. When no more edges can be added every face of an embedding is triangular. Every simple planar graph is thus a subgraph of such a *planar triangulation*.)

3.2. Demonstrate that every simple graph with $|E| < 9$ or with $n < 5$ is planar.

3.3. (*a*) Three houses have to be connected individually to the sources of three amenities (electricity, gas and water). Show that this cannot be done without at least two of the lines of supply crossing. (Because of this old problem, $K_{3,3}$ is sometimes known as the *amenities graph*.)

(*b*) Show that the Petersen graph (figure 6.14) contains a subgraph homeomorphic to $K_{3,3}$ and is therefore, according to Kuratowski's theorem, non-planar.

3.4. In a *completely regular* (simple planar) graph every vertex has the same degree $d(v)$, and every face has the same degree $d(f)$. Draw every completely regular (finite) graph. (For these graphs $2|E| = nd(v) = fd(f)$. Euler's formula then gives:

$$n = \frac{4d(f)}{2d(v) - d(f)\,(d(v)-2)}$$

For a fixed $d(v)$ we can find the allowable $d(f)$ consistent with a finite positive integer n. There are only five such graphs with $d(v) > 2$ and $d(f) > 2$.)

3.5. In the previous exercise we presumed that n was finite. Suppose, however, that $n = \infty$, then show that if G is completely regular and $d(v) > 2$ then $d(f)$ can only be 3, 4 or 6. This is a well-known fact in crystallography.

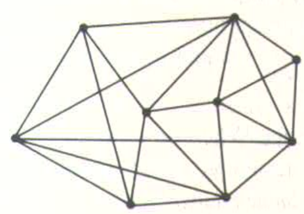
3.6. A *self-dual* is a simple planar graph which is isomorphic to its dual. Show, using Euler's formula, that if G is a self-dual then $2n = |E|+2$. How might a self dual be constructed for $n \geqslant 4$?

(Not every simple planar graph with $2n = |E|+2$ is a self-dual. Take care with vertices of degree 2.)

3.7. The *complement* $\bar{G}$ of a graph $G = (V, E)$ with n vertices is given by $\bar{G} = (K_n - E)$. Show that if $n \geqslant 11$, then at least one of G and $\bar{G}$ is non-planar.

(Use corollary 3.1. This result is also true for $n = 9$ and $n = 10$, but the proof is more difficult.)

3.8. Draw a planar embedding of the following graph in which every edge is a straight line.



(Every simple planar graph has an embedding in which each edge is a straight line, Fáry[17].)

3.9. Show that the average degree of the vertices in a simple planar graph is less than 6 (in fact less than or equal to $[6-(12/n)]$). Thus provide a different proof from that in the text that any simple planar graph must have at least one vertex of degree at most 5.

(Use corollary 3.1 and that the average degree of the vertices is $2|E|/n$.)

3.10. Show that if G_1 is a dual of G_2 and that if G_1' is 2-isomorphic to G_1, then G_1' is also a dual of G_2.
(Establish first that there is a one-to-one correspondence between edges of G_1 and edges of G_1' and that a circuit in G_1 is a circuit in G_1' and vice-versa. This exercise proves one-half of theorem 3.7, proof of the other half is quite lengthy – see Whitney[10].)

3.11. An electrical circuit consists of connections between two sets of terminals A and B. Set A has six and set B has five terminals. Each member of A is connected to every member of B. Show by construction that such a circuit can be printed on two sides of an insulating sheet with terminals extending through the sheet.
[In general the thickness of a complete bipartite graph $K_{r,s}$ is given by (see the chapter by White & Lowell in [7]):

$$T = \left\lfloor \frac{rs}{2(r+s)-4} \right\rfloor$$

There may be some rare exceptions to this formula, but none has less than 48 vertices.]

3.12. Find three planar graphs such that their union is the complete graph on ten vertices, K_{10}.

3.13. Embed the complete graph on seven vertices, K_7, on a torus.

3.14. Describe the details of an implementation of the planarity testing algorithm of figure 3.17 which is as efficient as you can make it.