SIAM J. ALG. DISC. METH.
Vol. 4, No. 3, September 1983

0196-5212/83/0403-0004 $01.25/0

CROSSING NUMBER IS NP-COMPLETE*

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Abstract. In this paper we consider a problem related to questions of optimal circuit layout: Given a graph or network, how can we embed it in a planar surface so as to minimize the number of edge-crossings? We show that this problem is NP-complete, and hence there is not likely to be any efficient way to design an *optimal* embedding.

A fundamental concept in graph theory is that of the *crossing number* $\nu(G)$ of a graph $G=(V,E)$. This is the least integer K such that G can be embedded in the plane so that there are no more than K pair-wise intersections of curves representing edges (not counting the required intersections at common endpoints). Recent work by Leighton [4] has shown that the crossing number of a graph can be used to obtain a lower bound on the amount of chip area required by that graph in a VLSI (very large scale integration) circuit layout, and the relevance of crossings to older technologies, such as printed circuits, has been discussed by Sinden [5].

There already exist efficient, linear-time algorithms for testing whether a graph has crossing number $\nu(G)=0$, i.e., for testing whether a graph is planar [3]. In this paper we show that the general CROSSING NUMBER decision problem "Given G and an integer K is $\nu(G)\leqq K$?" is NP-complete [1] and hence likely to be intractable. As a consequence, future research into crossing numbers will be justified in focusing on inexact methods that only *estimate* crossing numbers, and the quest for exact values of $\nu(G)$ will have to be restricted to promising special cases.

As defined, CROSSING NUMBER is in NP. One need only guess the K or fewer crossings (and the order in which they occur along edges involved in more than one crossing), create a new "crosspoint" vertex for each, replace each edge involved in one or more crossings by a path that contains all the crosspoint vertices associated with that edge in the appropriate order, and then test the resulting graph for planarity. Note that the above approach also allows us, for any fixed value of K, to test whether $\nu(G)\leqq K$ in polynomial time (the degree of the polynomial depending on K).

To prove that CROSSING NUMBER is NP-complete, we must show that a known NP-complete problem can be transformed to it. Our "known" NP-complete problem will be OPTIMAL LINEAR ARRANGEMENT [2]: "Given a graph $G=(V,E)$ and an integer K, is there a one-to-one function $f\colon V\to\{1,2,\cdots,|V|\}$ such that

$$\sum_{\{u,v\}\in E}|f(u)-f(v)|\leqq K?$$

We transform OPTIMAL LINEAR ARRANGEMENT to CROSSING NUMBER via an intermediate problem, which we shall call BIPARTITE CROSSING NUMBER: "Given a connected bipartite multigraph $G=(V_1,V_2,E)$ and an integer K, can G be embedded in a unit square so that all vertices of V_1 are on the northern boundary, all vertices in V_2 are on the southern boundary, all edges are within the square and there are at most K crossings?"

LEMMA 1. OPTIMAL LINEAR ARRANGEMENT $\propto$ BIPARTITE CROSSING NUMBER.

* Received by the editors March 29, 1982, and in revised form August 20, 1982.
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Proof. Suppose we are given an instance $G=(V,E)$, K of OPTIMAL LINEAR ARRANGEMENT, where $V=\{v_1, v_2, \cdots, v_n\}$. We may assume without loss of generality that G is connected. The corresponding instance of BIPARTITE CROSSING NUMBER is $G'=(V_1, V_2, E_1\cup E_2)$, K', where

$$V_1=\{u_i : 1\leqq i\leqq n\},$$

$$V_2=\{w_i : 1\leqq i\leqq n\},$$

$$E_1=\{|E|^2 \text{ copies of } \{u_i, w_i\} : 1\leqq i\leqq n\},$$

$$E_2=\{\{u_i, w_j\} : i<j \text{ and } \{v_i, v_j\}\in E\},$$

$$K'=|E|^2(K-|E|)+(|E|^2-1).$$

Note that both G' and K' are constructible in polynomial time, given G and K. Note also that G' is connected because G is. We must show that the answer for G, K is yes if and only if the answer for G', K' is also yes.

Suppose first that the desired ordering function f exists for G. Then we can construct the following layout of G'. Suppose the corners of the unit square have coordinates (0, 0), (0, 1), (1, 0) and (1, 1). We place each $u_i\in V_1$ at position $(1, f(v_i)/n)$ and each $w_i\in V_2$ at position $(0, f(v_i)/n)$, $1\leqq i\leqq n$. We then embed the multiple edges joining pairs $\{u_i, w_i\}$ so that none cross, as in Fig. 1. Each edge $\{u_i, w_j\}\in E_2$ will then cross $(|f(v_j)-f(v_i)|-1)\cdot|E|^2$ edges of E_1 and the total number of crossings of edges in E_1 with edges in E_2 will be at most

$$\sum_{\{u,v\}\in E}(|f(u)-f(v)|-1)\cdot|E|^2\leqq(K-|E|)\cdot|E|^2.$$

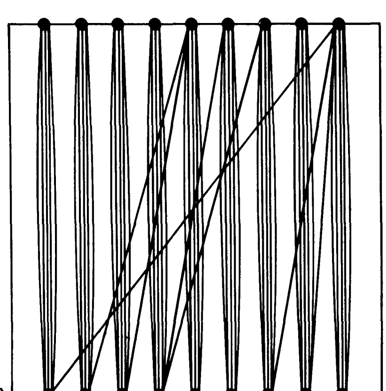

FIG. 1. *Embedding for Lemma* 1.

Since the total number of crossings between edges in E_2 is less than $(|E|^2-1)$, we conclude that the overall number of edge-crossings is at most K'.

Conversely, suppose the desired embedding of G' into the unit square exists. It naturally defines two one-to-one functions $f_1, f_2 : V\to\{1, 2, \cdots, |V|\}$ determined by the orderings of the vertices of V_1 and V_2 from left to right along their respective boundaries. These functions must be identical, since if $f_1(v_i)<f_1(v_j)$ and $f_2(v_i)>f_2(v_j)$, the embedding would contain at least $|E|^4$ crossings of edges $\{u_i, w_i\}$ with edges $\{u_j, w_j\}$, a contradiction of our bound on the number of crossings in the embedding. Thus the embedding looks like the one pictured in Fig. 1 and each edge $\{u_i, w_j\}\in E_2$ must be involved in at least $(|f_1(v_i)-f_1(v_j)|-1)\cdot|E|^2$ crossings. From this we conclude that

$$\sum_{\{u,v\}\in E}(|f_1(u)-f_1(v)|-1)\cdot|E|^2\leq K'=(K-|E|)\cdot|E|^2+(|E|^2-1),$$

which implies that

$$\sum_{\{u,v\}\in E}(|f_1(u)-f_1(v)|-1)\leq K-|E|,$$

and so f_1 will serve as the desired ordering for G. This completes the proof of the lemma. □

LEMMA 2. BIPARTITE CROSSING NUMBER $\propto$ CROSSING NUMBER.

Proof. We actually give a transformation to the version of CROSSING NUMBER where multigraphs are allowed. The final step to CROSSING NUMBER for graphs with no multiple edges allowed is obtained by simply adding a new degree-two vertex into the middle of each (multiple) edge, which eliminates the multiple edges without affecting the crossing number.

Suppose we are given an instance $G=(V_1, V_2, E), K$ of BIPARTITE CROSSING NUMBER. It is easy to construct the following multigraph $G'=(V', E\cup E_1\cup E_2\cup E_3)$ in polynomial time, where

$$V'=V_1\cup V_2\cup\{u_0, w_0\},$$
$$E_1=\{3K+1 \text{ copies of } \{u_0, u\}: u\in V_1\},$$
$$E_2=\{3K+1 \text{ copies of } \{w_0, w\}: w\in V_2\},$$
$$E_3=\{3K+1 \text{ copies of } \{u_0, w_0\}\}.$$

We claim that G has an embedding of the required form into the unit square (with K or fewer crossings) if and only if G' can be embedded in the plane with K or fewer crossings (the same K for both instances).

First, suppose the desired embedding of G into the unit square exists. Fig. 2 shows how the extra vertices and edges of G' can be added to the embedding (by being placed *outside* the unit square) with no increase in crossings.

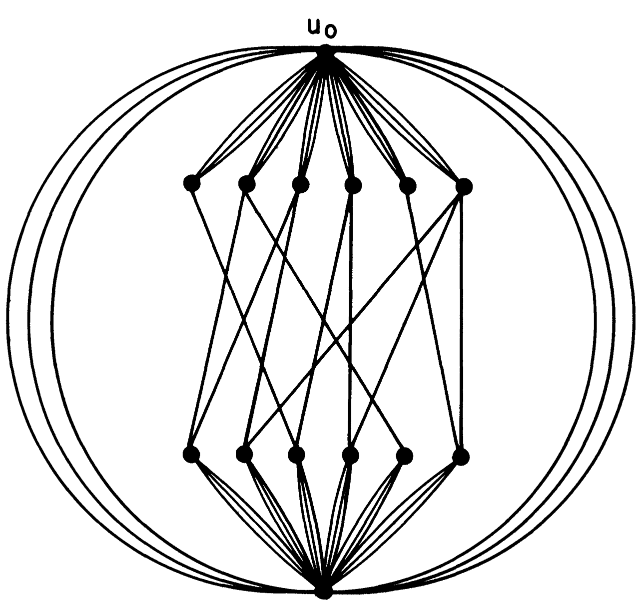

FIG. 2. *Embedding for Lemma* 2.

We now wish to argue that if the desired embedding of G' exists, there must be one whose form is just like that of Fig. 2. We proceed by a series of "normal form" simplifications.

Normalization 1. We may assume that each pair of edges crosses either 0 or 1 times and edges which share an endpoint do not cross at all. (This is easily proved using the transformation illustrated in Fig. 3, which always decreases the total number of crossings.) Thus each set of $3K+1$ multiple edges can be viewed as creating an ordered sequence of $3K$ bounded regions.

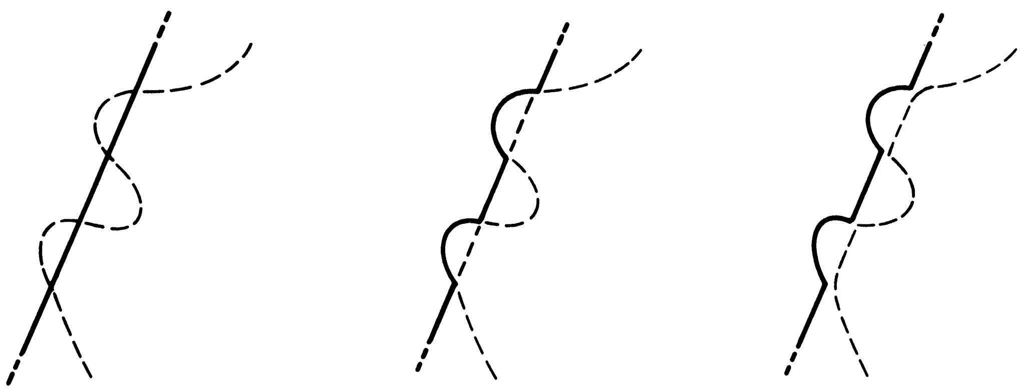

FIG. 3. *Removing multiple crossings.*

Normalization 2. The edges of E_1 divide the plane into a collection of regions, one of which is unbounded. By a standard transformation, we may assume that w_0 is inside (in the interior of) the unbounded region. Then, since each vertex in V_2 is connected by $3K+1$ edges to w_0, all these vertices must be inside the unbounded region too (if any such vertex were in a different region, it would introduce at least $3K+1$ crossings, which is too many).

Normalization 3. We may assume that no vertex is inside any of the $3K$ regions formed by the edges $(u_0, u\}$, for any fixed $u \in V_1$ and that no edge crosses any of these $3K+1$ edges. We may also assume that the same properties hold for the $3K$ regions formed by the edges $\{w_0, w\}$, for any fixed $w \in V_2$. We shall prove this for the case of $\{u_0, u\}$; the other case follows analogously.

From Normalization 2 none of the $3K$ regions bounded by edges $\{u_0, u\}$ can contain a vertex from $V_2 \cup \{w_0\}$. Thus an interior vertex, if it exists, must be from V_1. First let us make two observations about the middle K regions.

(a) No vertex from V_1 can be contained in any of the central K regions: such a vertex would have an edge to some vertex in V_2 since G is connected and that edge would have at least $K+1$ crossings.

(b) No edge can cross any of the K middle regions: such an edge would have to cross *all* K regions if it crossed any, since by Normalization 1 it cannot double back, and by (a) its end-points must be at least K regions (and hence $K+1$ boundary edges) apart.

Given (a) and (b), it follows that we can transform the embedding, by moving all edges joining u_0 and u into the interior of a single one of the middle regions, and no new crossing will be created. As a result, all vertices other than u_0 and u are left on the outside, and no edge will cross any of the boundaries.

Note that at this point we have obtained an embedding which is topologically equivalent to one like that in Fig. 2, except possibly for the edges in the original set E and the $3K+1$ edges joining u_0 to w_0.

Normalization 4. We may assume that all of the vertices in $V_1 \cup V_2$ and all of the edges in E are contained inside the same one of the $3K$ bounded regions formed by the edges joining u_0 to w_0.

Let us begin our proof of this claim by numbering the bounded regions in order, R_1 through R_{3K}, with R_0 being the unbounded region. Suppose there is a vertex v inside region R_I. Then there can be no vertex v' in regions $R_{I+K+1(\mathrm{mod}\,3K+1)}$ through $R_{I+2K(\mathrm{mod}\,3K+1)}$. This is because there was a path in our original graph from v to v', and this path would have to cross at least $K+1$ of the edges $\{u_0, w_0\}$ if v' were in one of the prescribed regions. Consequently, as in claim (b) of the proof of Normalization 3, there can be no edges passing through any of these K regions. Thus, as in Normalization 3 we can move all the edges joining u_0 to w_0 into just one of these empty regions, without creating any new crossings. This leaves V_1, V_2 and all of E in the single unbounded region. Now a simple transformation sends our embedding to one in which all of G is contained within the same *bounded region*.

Finalization. At this point we are done with the proof of Lemma 2, for the embedding created by our four normalizations is now topologically equivalent to one in the form of Fig. 2 and hence induces the desired embedding of G into the unit square. □

The main theorem of this paper (and its title) follow as an immediate consequence of Lemmas 1 and 2.

Acknowledgment. The authors thank Gary Miller and Tom Leighton for suggestions that improved the presentation of this result.

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