Апроксимизация на Стирлинг: n! $= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{1}{51840n^3} + ...\right)$

Сл. n! $\approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Пр. 1

log(n!) ≈ log($\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$)=log($\sqrt{2\pi}$) + $\log\left(n^{\frac{1}{2}}\right) + \log(n^n) + \log(e^{-n})$=const+$\frac{\log(n)}{2}$+nlog(n)-nlog(e) ⇒

log(n!) ≍ nlog(n)

Пр. 2

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \approx \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2} = \frac{2\sqrt{\pi n}\left(\frac{2n}{e}\right)^{2n}}{2\pi n\left(\frac{n}{e}\right)^{2n}} = \frac{2\sqrt{\pi n}\, 2^{2n}\left(\frac{n}{e}\right)^{2n}}{2\pi n\left(\frac{n}{e}\right)^{2n}} = \frac{\sqrt{\pi n}\, 4^n}{\pi n} = \frac{4^n}{\sqrt{\pi n}}$$

$$\Rightarrow \binom{2n}{n} \asymp \frac{4^n}{\sqrt{\pi n}}$$

Зад. 6 (от миналия път)

$n^n \succ n! \succ a^n \succ n^{\log(n)} \succ n^3 \succ n^2 \succ \sqrt{n} \succ \log^2(n) \succ \log(n) \succ \log^{(2)}(n)$≻a≻$\frac{1}{n^2}$

1. $n^n \succ n!$

Тв. $\forall n \in \mathbb{N}^+$ е изп. $0 \le n! \le n^{n-1}$

Док. по индукция..

База: n=1 $0 \le 1! \le 1^0$ -ок е

ИП: Нека е изп. за n=k т.е е изп. $0 \le k! \le k^{k-1}$

ИС: Ще док., че е изп. за n=k+1.. т.е ще док., че е вярно $0 \le (k+1)! \le (k+1)^k$

Използваме ИП: $0 \le k! \le k^{k-1} \le (k+1)^{k-1}$.. умножаваме по k+1 всяка страна и получаваме

$0 \le (k+1)! \le (k+1)\, k^{k-1} \le (k+1)^k$.. т.е док. исканото

⇒Тв. е доказано

Сега ще използваме Тв. $0 \le n! \le n^{n-1}$.. сега, делим от 3те страни с n^n и получавме

0≤$\frac{n!}{n^n} \le \frac{1}{n}$.. сега от лемата за 2мата полицаи получаваме, че $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

2. n! ? a^n

log(n!) ? log(a^n)

nlog(n) ? nlog(a)

$\lim_{n\to\infty} \frac{\text{nlog}(a)}{\text{nlog}(n)} = 0$

⇒ log(n!) ≻ log(a^n) ⇒ n! ≻ a^n

3. a^n ? $n^{\log(n)}$

log(a^n) ? $\log\left(n^{\log(n)}\right)$

nlog(a) ? $\log^2$(n)

⇒ от св.10 ⇒ log(a^n) ≻ $\log\left(n^{\log(n)}\right)$ ⇒ $a^n \succ n^{\log(n)}$

9. log(n) ? $\log^{(2)}(n)$

$\lim_{n->\infty} \frac{\log(\log(n))}{\log(n)}$ = Лопитал = $\lim_{n\to\infty} \frac{\frac{1}{\log(n)}(\log(n))'}{(\log(n))'} = \lim_{n\to\infty} \frac{1}{\log(n)} = 0$

Лопитал: Ако имаме, че $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\infty}{\infty}$, то имаме че $\lim_{n\to\infty} \frac{f(n)}{g(n)}$=$\lim_{n\to\infty} \frac{f'(n)}{g'(n)}$

Зад. 1

Да се док., че $\sqrt[n]{n} \asymp 1$

Лема: $\lim_{x\to\infty} f(x) = a > 0 \Leftrightarrow \lim_{x\to\infty} \ln(f(x)) = \ln(a)$

Док. (изображение 1)

$\lim_{n\to\infty} \ln\left(\sqrt[n]{n}\right) = \lim_{n\to\infty} \ln\left(n^{\frac{1}{n}}\right) = \lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

От лемата $\Rightarrow \lim_{n\to\infty} \sqrt[n]{n} = e^0 = 1 \Rightarrow \sqrt[n]{n} \asymp 1$

Зад. 2

Да се подредят по асимптотика следните ф-ии:

$\sqrt{2}^{\log(n)}$, n^3, $n!$, $(\log(n))!$, $\log^2(n)$, $\log(n!)$, 2^{2^n}, $n^{\frac{1}{\log(n)}}$, $\log(\log(n))$, $\left(\frac{3}{2}\right)^n$,

$n2^n$, $4^{\log(n)}$, $(n+1)!$, $\sqrt{\log(n)}$, $2^{\sqrt{2\log(n)}}$, $n^{\log(\log(n))}$, $\log(n)$, $2^{\log(n)}$, $(\log(n))^{\log(n)}$

Подредбата

е:

$n^{\frac{1}{\log(n)}} \prec \log(\log(n)) \prec \sqrt{\log(n)} \prec \log(n) \prec \log^2(n) \prec 2^{\sqrt{2\log(n)}} \prec \sqrt{2}^{\log(n)} \prec 2^{\log(n)} \prec$

$\log(n!) \prec 4^{\log(n)} \prec n^3 \prec (\log(n))! \prec (\log(n))^{\log(n)} \asymp n^{\log(\log(n))} \prec \left(\frac{3}{2}\right)^n \prec n2^n \prec n! \prec (n+1)! \prec 2^{2^n}$

1. $n^{\frac{1}{\log_2(n)}} = n^{\frac{\log_2(2)}{\log_2(n)}} = n^{\log_n 2} = 2$

2. $\log(\log(n))$? $\sqrt{\log(n)}$

$\log(\log(\log(n)))$? $\frac{\log(\log(n))}{2}$

От лопитал $\Rightarrow \lim_{n\to\infty} \frac{2\log(\log(\log(n)))}{\log(\log(n))} = 0$

$\Rightarrow \log(\log(n)) \prec \sqrt{\log(n)}$

3. $\lim_{n\to\infty} \frac{\sqrt{\log(n)}}{\log(n)} = 0$

4. $\lim_{n\to\infty} \frac{\log(n)}{(\log(n))^2} = 0$

5. $\log^2(n)$? $2^{\sqrt{2\log(n)}}$

$\log\left((\log(n))^2\right)$? $\sqrt{2\log(n)}\ \log2$

$2\log(\log(n))$? $\sqrt{2}\ \sqrt{\log(n)}$

аналог. на 2. $\Rightarrow \log^2(n) \prec 2^{\sqrt{2\log(n)}}$

6. $\sqrt{2}^{\log(n)} = \left(2^{\frac{1}{2}}\right)^{\log(n)} = 2^{\frac{\log(n)}{2}} = 2^{\log\left(\sqrt{n}\right)} = \left(\sqrt{n}\right)^{\log(2)} = \left(\sqrt{n}\right)^1 = \sqrt{n}$

$2^{\sqrt{2\log(n)}}$? $\sqrt{n}$

$\sqrt{2\log(n)}$? $\frac{\log(n)}{2}$

аналог. на 3. $\Rightarrow \sqrt{2\log(n)} \prec \sqrt{n} = \sqrt{2}^{\log(n)}$

7. $2^{\log(n)} = n^{\log(2)} = n$

$\sqrt{n} \prec n$

т.е $\sqrt{2}^{\log(n)} \prec 2^{\log(n)}$

8. $\log(n!) \asymp n\log(n)$

т.е имаме, че $n \prec n\log(n)$

$\Rightarrow 2^{\log(n)} \prec \log(n!)$

9. $4^{\log(n)} = n^{\log(4)} = n^2$

т.е имаме, че $n\log(n) \prec n^2$

$\Rightarrow \log(n!) \prec 4^{\log(n)}$

10. Знаем, че $n^2 \prec n^3$

$\Rightarrow 4^{\log(n)} \prec n^3$

11. n^3 ? $(\log(n))!$

$\log\left(n^3\right)$? $\log((\log(n))!)$

$3\log(n)$? $\log(n)*\log(\log(n))$

$\Rightarrow n^3 \prec (\log(n))!$

12. $(\log(n))!$? $(\log(n))^{\log(n)}$

Пол. $m=\log(n)$

$m!$? m^m

това го док. по-нагоре $\Rightarrow m! \prec m^m \Rightarrow (\log(n))! \prec (\log(n))^{\log(n)}$

13. $(\log(n))^{\log(n)} = n^{\log(\log(n))} \Rightarrow (\log(n))^{\log(n)} \asymp n^{\log(\log(n))}$

14. $n^{\log(\log(n))}$? $\left(\frac{3}{2}\right)^n$

$\log\left(n^{\log(\log(n))}\right)$? $n\log\left(\frac{3}{2}\right)$

$\log(\log(n))\log(n)$? $n\log\left(\frac{3}{2}\right)$

$\log(\log(n))\log(n) \prec \log(n)\log(n) \prec n\log\left(\frac{3}{2}\right)$

15. $\left(\frac{3}{2}\right)^n \prec 2^n \prec n2^n$

16. $n2^n$? $n!$

$\log(n2^n)$? $n\log(n)$

$\log(n) + \log(2^n)$? $n\log(n)$

$\log(n) + n$? $n\log(n)$

$\Rightarrow n2^n \prec n!$

17. $\lim_{n\to\infty} \frac{n!}{(n+1)\, n!}=0$

18. $(n+1)!$? 2^{2^n}

$\log((n+1)!)$? $2^n\log(2)$

$(n+1)\log(n+1)$? 2^n

$(n+1)\log(n+1) \prec (n+1)^2 \prec 2^n$