Зад. 2 Подредете по асимптотично нарастване следните шест функции.

$$f_1(n) = \sum_{k=1}^{n} \binom{n}{k} \qquad f_2(n) = \sum_{k=1}^{n} \binom{k}{n} \qquad f_3(n) = \sum_{k=1}^{n} \binom{2n}{k}$$

$$f_4(n) = \sum_{k=1}^{n!} \frac{1}{2^k} \qquad f_5(n) = \sum_{k=1}^{n} \frac{1}{k^2} \qquad f_6(n) = \sum_{k=1}^{n} \frac{1}{k}$$

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \ldots$$

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + .. + \binom{n}{n}$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + .. + \binom{n}{n}$$

$$f_1(n) = \sum_{1}^{n} \binom{n}{k} = \sum_{0}^{n} \binom{n}{k} - 1 = 2^n - 1 \asymp 2^n$$

$$f_2(n) = \sum_{1}^{n} \binom{k}{n} = \binom{0}{n} + \binom{0}{n} + \ldots + \binom{n}{n} = 0 + 0 + \ldots + 1 = 1$$

$$f_3(n) = \sum_{1}^{n} \binom{2n}{k} = \binom{2n}{1} + \binom{2n}{2} + .. + \binom{2n}{n}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$$\binom{n}{n-k} = \frac{n!}{(n-n+k)!(n-k)!} = \frac{n!}{(n-k)!k!}$$

$$f_3(n) = \sum_{1}^{n} \binom{2n}{k} = \sum_{0}^{n} \binom{2n}{k} - 1 = \frac{2^{2n}}{2} - 1 = \frac{1}{2}4^n - 1 \asymp 4^n$$

$$f_3(n) = \sum_{1}^{n} \binom{2n}{k} > \binom{2n}{n}$$

$$\binom{2n}{n} = \frac{(2n)!}{(2n-n)!n!} = \frac{(2n)!}{n!n!} \asymp \frac{\sqrt{4\pi n}\frac{(2n)^{2n}}{e^{2n}}}{\left(\sqrt{2\pi n}\frac{n^n}{e^n}\right)^2}$$

$$= \frac{\sqrt{4\pi n}\frac{2^{2n}n^{2n}}{e^{2n}}}{2\pi n\frac{n^{2n}}{e^{2n}}} = \frac{\sqrt{\pi n}\,2^{2n}}{\pi n} = 4^n\frac{\sqrt{\pi n}}{\pi n}$$

$$f_4(n) = \sum_1^{n!}\frac{1}{2^k} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+.... \leqslant \sum_0^{\infty}\frac{1}{2^k} = \frac{1}{1-1/2} = 2 \asymp 1$$

$$f_5(n) = \sum_1^{n}\frac{1}{k^2} \asymp 1$$

$$f_6(n) = \sum_1^{n}\frac{1}{k} \asymp ln\ n$$

$$f_2 \asymp f_4 \asymp f_5 \prec f_6 \prec f_1 \prec f_3$$

$$ln\ n \asymp log_2 n$$

$$1 \prec lg\ lg\ n \prec lg\ n \prec (lg\ n)^2 \prec \sqrt{n} \prec n \prec n\ lg\ lg\ n \prec n\ lg\ n \prec n\sqrt{n} \prec$$

$$n^2 \prec n^2\ lg\ lg\ n \prec n^2\ lg\ n$$

$$2^n \prec 2^n\sqrt{n} \prec 2^n\ n^2$$

$$2^n \prec e^n \prec 3^n \prec n! \prec (n+1)! \prec n^n \asymp (n+1)^n \prec n^{n+1} \prec (2n)! \prec n^{n^2}$$

$$\frac{n!}{(n+1)!} = \frac{n!}{n!(n+1)} = \frac{1}{n+1}$$

$$n! \asymp \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$(n+1)! \prec\ n^n$$

$$(n+1)! \asymp \sqrt{2\pi(n+1)}\left(\frac{(n+1)}{e}\right)^{n+1}$$

$$\lim_{n\to\infty} \frac{\sqrt{2\pi(n+1)}\left(\frac{(n+1)}{e}\right)^{n+1}}{n^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi(n+1)}\frac{(n+1)^{n+1}}{e^{n+1}}}{n^n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{2\pi(n+1)}(n+1)^n(n+1)}{n^n e^{n+1}} = \lim_{n\to\infty} \frac{\sqrt{2\pi(n+1)}e(n+1)}{e^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{2\pi(n+1)}\ (n+1)}{e^n} = 0$$

$$n^n \asymp (n+1)^n$$

$$\lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{1}{e}$$

$$(n+1)^n \prec n^{n+1}$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n}\frac{1}{n} = \lim_{n\to\infty} \frac{e}{n} = 0$$

$$n^{n+1} \prec (2n)!$$

$$(2n)!\ \asymp \sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}$$

$$\lim_{n\to\infty} \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{n^{n+1}} = \lim_{n\to\infty} \frac{\sqrt{4\pi n}\ (2n)^{2n}}{n^{n+1}\ e^{2n}} = \lim_{n\to\infty} \frac{\sqrt{4\pi n}\ 4^n n^{2n}}{n^n\ n\ e^{2n}}$$

$$= \lim_{n\to\infty} \sqrt{4\pi n}\ \frac{4^n}{n}\frac{e^{ln\ n * n}}{e^{2n}} \ = ?$$

$$\lim_{n \to \infty} \frac{e^{\ln n * n}}{e^{2n}} = \lim_{n \to \infty} e^{n(\ln n - 2)} = e^{\lim_{n \to \infty} n(\ln n - 2)} = e^{\lim_{n \to \infty} n * \lim_{n \to \infty}(\ln n - 2)} = \infty$$

$$\lim_{n \to \infty} \frac{4^n}{n} = \lim_{n \to \infty} \frac{\ln 4 * 4^n}{1} = \infty$$

$$\lim_{n \to \infty} \sqrt{4\pi n} = \infty$$

$$\implies \lim_{n \to \infty} \sqrt{4\pi n}\, \frac{4^n}{n} \frac{e^{\ln n * n}}{e^{2n}} = \infty$$

$$(2n)! \prec n^{n^2}$$

$$lg((2n)!) \prec lg\left(n^{n^2}\right) \implies (2n)! \prec n^{n^2}$$

$$lg((2n)!) \asymp 2n\, lg\, 2n \asymp n\, lg\, n$$

$$lg\left(n^{n^2}\right) = n^2 lgn$$

$$nlgn \prec n^2 lgn$$

```
int sum(int a[n])
{
  int s = 0;
  // Invariant: Преди всяка проверка i <= n е вярно, че s =
a[1]+...+a[i-1]
  for (int i = 1; i <= n; i ++) {
    s = s + a[i];
  }
  return s;
}
```

База

```
s=0
i=1
```

```
1<=n ? s=a[1]+...+a[i-1]
```

Поддръжка

```
s=a[1]+...+a[i-1]
s=s+a[i]=a[1]+...+a[i-1]+a[i]

i <= n ? s=a[1]+...+a[i-1]
```

Завършек

```
i=n+1
s=a[1]+...+a[i-1]=a[1]+...+a[n+1-1]=a[1]+...+a[n]



int max(int a[n])
{
  int m = a[1];
  // преди проверката k<=n, m = max(a[1],...,a[k-1])
  for (int i = 2; i <= n; i ++) {
      if(a[i] > m) {
        m = a[i];
      }
  }
  return m;
}
```

Поддръжка

```
i<=m m=max(a[1],...,a[i-1])
 if(a[i] > m) {
    m = a[i];
 }
```

База

```
m=a[1], i=2,
```

2<=n преди това вярно ли е m=max(a[1],...,a[i-1])=max(a[1]) вярно е защото m=a[1]