Апроксимация на Стирлинг:

$n! = \sqrt{2\pi \mathrm{n}}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}+\ldots\right)$

$n! \approx \sqrt{2\pi \mathrm{n}}\left(\frac{n}{e}\right)^n$

Приложения:

$\cdot \log(n!) \approx \log\left(\sqrt{2\pi \mathrm{n}}\left(\frac{n}{e}\right)^n\right) = \log\left(\sqrt{2\pi \mathrm{n}}\right) + \log\left(\left(\frac{n}{e}\right)^n\right) =$

$\log\left(\sqrt{2\pi}\right) + \log\left(\sqrt{n}\right) + \log(n^n) - \log(e^n) = \log(\mathrm{const}) + \frac{1}{2}\log(n) + n\log(n) - n\log(e) = O(n\log(n))$

$\cdot \binom{2n}{n} = \frac{(2n)!}{n!\,n!} \approx \frac{\sqrt{4\pi \mathrm{n}}\left(\frac{2n}{e}\right)^{2n}}{2\pi \mathrm{n}\left(\frac{n}{e}\right)^{2n}} = \frac{2^{2n}n^{2n}}{\sqrt{\pi \mathrm{n}}\;n^{2n}} = \frac{2^{2n}}{\sqrt{\pi \mathrm{n}}} = \frac{4^n}{\sqrt{\pi \mathrm{n}}}$

Зад. 6 (миналия път)

$n^n \succ n! \succ a^n \succ n^{\log(n)} \succ n^3 \succ n^2 \succ \sqrt{n} \succ \log^2(n) \succ \log(n) \succ \log^{(2)}(n) \succ a \succ n^{-2}$

1. $n^n \succ n!$ - от ДИС

2. $n! \;?\; a^n$

$\log(n!) \;?\; \log(a^n)$

$n\log(n) \;?\; n\log(a)$

$\Rightarrow n! \succ a^n$

3. $a^n \;?\; n^{\log(n)}$

$\log(a^n) \;?\; \log\left(n^{\log(n)}\right)$

$\mathrm{n}\log(a) \;?\; (\log(n))^2$

$\overset{\text{св}.10}{\Rightarrow} a^n \succ n^{\log(n)}$

4. $n^{\log(n)} \;?\; n^3$

$\log\left(n^{\log(n)}\right) \;?\; \log\left(n^3\right)$

$(\log(n))^2 \;?\; 3\log(n)$

$\lim_{n\to\infty}\frac{(\log(n))}{(\log(n))^2} = 0 \Rightarrow n^3 = o\left(n^{\log(n)}\right) \Rightarrow n^{\log(n)} \succ n^3$

9. $\log(n) \;?\; \log^{(2)}(n)$

Пол. $m = \log(n)$

$m \;?\; \log(m)$

$\Rightarrow \log(n) \succ \log^{(2)}(n)$

Алтернативен начин: $\lim_{n\to\infty}\frac{\log^{(2)}(n)}{\log(n)} = \lim_{n\to\infty}\frac{(\log(\log(n)))'}{(\log(n))'} = \lim_{n\to\infty}\frac{1}{\ln 2 * \log(n)} * \frac{(\log(n))'}{(\log(n))'} = 0$

10. $a \;?\; n^{-2}$

$\lim_{n\to\infty}\frac{n^{-2}}{a} = 0$

$\Rightarrow a \succ n^{-2}$

Лема
$\lim_{x\to\infty} f(x) = a > 0 \Leftrightarrow \lim_{x\to\infty} \log(f(x)) = \log(a)$

Док.
Изображение 1 (и още ДИС)

Зад. 1
Да се докаже, че $\sqrt[n]{n} \asymp 1$

Решение:
$\lim_{x\to\infty} \ln\left(\sqrt[n]{n}\right) = \lim_{x\to\infty} \frac{1}{n} \ln(n) = 0$
$0 = \lim_{x\to\infty} \ln\left(\sqrt[n]{n}\right) \overset{\text{Лема}}{=} \ln\left(\lim_{x\to\infty}\left(\sqrt[n]{n}\right)\right)$
$\Rightarrow \lim_{x\to\infty}\left(\sqrt[n]{n}\right) = 1$

Зад. 2
$\left(\sqrt{2}\right)^{\log(n)}$, n^3, $n!$, $(\log(n))!$, $\log^2(n)$, $\log(n!)$, 2^{2^n}, $n^{\frac{1}{\log(n)}}$, $\log^{(2)}(n)$, $\left(\frac{3}{2}\right)^n$, $n2^n$,
$4^{\log(n)}$, $(n+1)!$, $\sqrt{\log(n)}$, $2^{\sqrt{2\log(n)}}$, $n^{\log(\log(n))}$, $\log(n)$, $2^{\log(n)}$, $(\log(n))^{\log(n)}$

Решение:
Ще докажем, че реда е :
$n^{\frac{1}{\log(n)}} \prec \log^{(2)}(n) \prec \sqrt{\log(n)} \prec \log(n) \prec \log^2(n) \prec 2^{\sqrt{2\log(n)}} \prec \left(\sqrt{2}\right)^{\log(n)} \prec 2^{\log(n)} \prec$
$\log(n!) \prec 4^{\log(n)} \prec n^3 \prec (\log(n))! \prec (\log(n))^{\log(n)} \asymp n^{\log(\log(n))} \prec \left(\frac{3}{2}\right)^n \prec n2^n \prec n! \prec (n+1)! \prec 2^{2^n}$

1. $n^{\frac{1}{\log(n)}}$? $\log^{(2)}(n)$
$n^{\frac{1}{\log_2(n)}} = n^{\log_n(2)} = 2^{\log_n(n)} = 2^1$
$\Rightarrow n^{\frac{1}{\log(n)}} \prec \log^{(2)}(n)$

2. $\log(\log(n))$? $(\log(n))^{\frac{1}{2}}$
Пол. $m = \log(n)$
$\log(m)$? $m^{\frac{1}{2}}$
$\Rightarrow \log(\log(n)) \prec (\log(n))^{\frac{1}{2}}$

3. $(\log(n))^{\frac{1}{2}}$? $\log(n)$
$\lim_{n\to\infty} \frac{\sqrt{\log(n)}}{\log(n)} = 0$
$\Rightarrow (\log(n))^{\frac{1}{2}} \prec \log(n)$

4. $\log(n)$? $\log^2(n)$
$\lim_{n\to\infty} \frac{\log(n)}{\log^2(n)} = 0$
$\Rightarrow \log(n) \prec \log^2(n)$

5. $\log^2(n)$? $2^{\sqrt{2\log(n)}}$
$\log\left((\log(n))^2\right)$? $\log\left(2^{\sqrt{2\log(n)}}\right)$
$2\log(\log(n))$? $\sqrt{2\log(n)}$
Пол. $m = \log(n)$
$2\log(m)$? $\sqrt{2m}$
$\Rightarrow \log^2(n) \prec 2^{\sqrt{2\log(n)}}$

6. $\mathbf{2^{\sqrt{2\log(n)}}\ ?\ \left(\sqrt{2}\right)^{\log(n)}}$

$\left(\sqrt{2}\right)^{\log(n)} = 2^{\frac{\log(n)}{2}} = 2^{\log\left(\sqrt{n}\right)} = \left(\sqrt{n}\right)^{\log(2)} = \sqrt{n}$

$2^{\sqrt{2\log(n)}}\ ?\ \sqrt{n}$

$\log\left(2^{\sqrt{2\log(n)}}\right)\ ?\ \log\left(\sqrt{n}\right)$

$\sqrt{2\log(n)}\ ?\ \frac{1}{2}\log(n)$

$2^{\sqrt{2\log(n)}} \prec \left(\sqrt{2}\right)^{\log(n)}$

7. $\mathbf{\left(\sqrt{2}\right)^{\log(n)}\ ?\ 2^{\log(n)}}$

$2^{\frac{\log(n)}{2}}\ ?\ 2^{\log(n)}$

$\lim_{n\to\infty}\frac{2^{\frac{\log(n)}{2}}}{2^{\log(n)}} = \lim_{n\to\infty}\frac{1}{2^{\frac{\log(n)}{2}}} = 0$

$\Rightarrow \left(\sqrt{2}\right)^{\log(n)} \prec 2^{\log(n)}$

8. $\mathbf{2^{\log(n)}\ ?\ \log(n!)}$

$2^{\log(n)}\ ?\ \text{nlog}(n)$

$n\ ?\ \text{nlog}(n)$

$2^{\log(n)} \prec \log(n!)$

9. $\mathbf{\log(n!)\ ?\ 4^{\log(n)}}$

$\text{nlog}(n)\ ?\ n^2$

$\Rightarrow \log(n!) \prec 4^{\log(n)}$

10. $\mathbf{4^{\log(n)}\ ?\ n^3}$

$n^2\ ?\ n^3$

$\Rightarrow 4^{\log(n)} \prec n^3$

11. $\mathbf{n^3\ ?\ (\log(n))!}$

$\log\left(n^3\right)\ ?\ \log((\text{logn})!)$

$(\log(n))! = \sqrt{2\pi(\log(n))}\left(\frac{\log(n)}{e}\right)^{\log(n)}$

тоест имаме : $\log((\log(n))!) = \log\left((\log(n))^{\log(n)}\right) = \log(n) * \log(\log(n))$

$3\log(n)\ ?\ \log(n) * \log(\log(n))$

$\lim_{n\to\infty}\frac{3\log(n)}{\log(n)*\log(\log(n))} = 0$

$\Rightarrow n^3 \prec (\log(n))!$

12. $\mathbf{(\log(n))!\ ?\ (\log(n))^{\log(n)}}$

Пол. $m = \log(n)$

$m!\ ?\ m^m$

$\Rightarrow (\log(n))! \prec (\log(n))^{\log(n)}$

13. $\mathbf{(\log(n))^{\log(n)}\ ?\ n^{\log(\log(n))}}$

$\log\left((\log(n))^{\log(n)}\right) ?\ \log\left(n^{\log(\log(n))}\right)$

$\log(n) * \log(\log(n))\ ?\ \log(\log(n)) * \log(n)$

$\Rightarrow (\log(n))^{\log(n)} \asymp n^{\log(\log(n))}$

14. $\mathbf{n^{\log(\log(n))}\ ?\ \left(\frac{3}{2}\right)^n}$

$\log\left(n^{\log(\log(n))}\right)\ ?\ \log\left(\left(\frac{3}{2}\right)^n\right)$

$\log(\log(n)) * \log(n)\ ?\ n * \log\left(\frac{3}{2}\right)$

знаем, че $\log(\log(n)) * \log(n) \prec \log^2(n) \prec n$

$\Rightarrow n^{\log(\log(n))} \prec \left(\frac{3}{2}\right)^n$

15. $\left(\frac{3}{2}\right)^n$ **?** $\mathbf{n2^n}$

Знаем, че $\left(\frac{3}{2}\right)^n \prec 2^n \prec \text{n}2^n$, защото

$\lim_{n\to\infty} \frac{\left(\frac{3}{2}\right)^n}{2^n} = \lim_{n\to\infty} \frac{3^n}{4^n} = \lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0$

$\Rightarrow \left(\frac{3}{2}\right)^n \prec \text{n}2^n$

16. $\boldsymbol{n2^n}$ **?** $\boldsymbol{n!}$

$\log(n2^n)$? $n\log(n)$

$\log(n) + n$? $n\log(n)$

$\Rightarrow n2^n \prec n!$

17. $\boldsymbol{n!}$ **?** $\boldsymbol{(n+1)!}$

$\lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{n!\,(n+1)} = 0$

$\Rightarrow n! \prec (n+1)!$

18. $\boldsymbol{(n+1)!}$ **?** $\mathbf{2^{2^n}}$

$\log((n+1)!)$? $\log\left(2^{2^n}\right)$

$\log\left((n+1)^{n+1}\right)$? 2^n

$(n+1)\log(n+1)$? 2^n

Знаем, че $(n+1)\log(n+1) \prec (n+1)^2 \prec 2^{n+1} \asymp 2^n$

$\Rightarrow (n+1)! \prec 2^{2^n}$