Hypercubes

Problem 1. *Let B^n be an n-dimensional hypercube graph. Let A be a subset of the vertices of B^n such that $|A| > 2^{n-1}$. Let H be the subgraph of B^n induced by A. Prove that H has at least n edges.*

Solution:
We use the notation $\{\ \ \}_M$ for multisets as in $\{1,2,2,3,3,3\}_M$. We identify the vertex set of B^n with the set of all binary vectors of length n. Let $m = |A|$ and $A = \{\widetilde{\alpha_1}, \widetilde{\alpha_2}, \ldots, \widetilde{\alpha_m}\}$. Let $\widetilde{\alpha_i} = \left(\alpha_i^1, \alpha_i^2, \ldots, \alpha_i^n\right)$ for $1 \leq i \leq m$, where $\alpha_i^j \in \{0,1\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. For every k such that $1 \leq k \leq n$, define that *the* k*-th contraction of* A is the multiset of vectors obtained from A after removing the k-th position of each vector. We denote the k-th contraction of A by "$A|_k$". Formally, $A|_k = \{\widetilde{\alpha_1}|_k, \widetilde{\alpha_2}|_k, \ldots, \widetilde{\alpha_m}|_k\}_M$ where

$$\widetilde{\alpha_i}|_k = \left(\alpha_i^1, \alpha_i^2, \ldots, \alpha_i^{k-1}, \alpha_i^{k+1}, \ldots, \alpha_i^n\right) \text{ for } 1 \leq i \leq m.$$

Clearly, the vectors in $A|_k$ have length $n-1$. There can be at most 2^{n-1} distinct binary vectors of length n. But by construction $A|_k$ has $m > 2^{n-1}$ vectors, therefore at least two vectors in $A|_k$ are the same. Let $\widetilde{\alpha_{p_k}}|_k$ and $\widetilde{\alpha_{q_k}}|_k$ be any two vectors in $A|_k$ that are the same for some indices p_k and q_k such that $1 \leq p_k < q_k \leq m$. The indices are themselves indexed by k because they depend on the value of k.

Note that A is not a multiset but a normal set, *i.e.* without repeating elements. Therefore it must be the case that $\widetilde{\alpha_{p_k}} \neq \widetilde{\alpha_{q_k}}$. But $\widetilde{\alpha_{p_k}}$ and $\widetilde{\alpha_{q_k}}$ can differ only in the k-th position. Therefore, by the definition of the hypercube graph, for every k such that $1 \leq k \leq n$, there is an edge $e_k = (\widetilde{\alpha_{p_k}}, \widetilde{\alpha_{q_k}})$ both in B^n and H.

Next we argue that for any two distinct values of k, say s and t, the edges e_s and e_t are distinct. Assume the opposite:

$$(\widetilde{\alpha_{p_s}}, \widetilde{\alpha_{q_s}}) = (\widetilde{\alpha_{p_t}}, \widetilde{\alpha_{q_t}}) \text{ for some } s, t, \text{ such that } 1 \leq s < t \leq n \tag{1}$$

Recall that $\widetilde{\alpha_{p_s}}$ and $\widetilde{\alpha_{q_s}}$ differ in precisely one position, namely the s-th position. Therefore,

$$\widetilde{\alpha_{p_s}} = \beta_s, 0, \gamma_s \text{ and} \tag{2}$$
$$\widetilde{\alpha_{q_s}} = \beta_s, 1, \gamma_s \tag{3}$$

or

$$\widetilde{\alpha_{p_s}} = \beta_s, 1, \gamma_s \text{ and} \tag{4}$$
$$\widetilde{\alpha_{q_s}} = \beta_s, 0, \gamma_s \tag{5}$$

where β_s and γ_s are binary vectors such that $|\beta_s| + |\gamma_s| = n - 1$. Likewise,

$\widetilde{\alpha_{p_t}} = \beta_t, 0, \gamma_t$ and
$\widetilde{\alpha_{q_t}} = \beta_t, 1, \gamma_t$

or

$\widetilde{\alpha_{p_t}} = \beta_t, 1, \gamma_t$ and
$\widetilde{\alpha_{q_t}} = \beta_t, 0, \gamma_t$

where β_t and γ_t are binary vectors such that $|\beta_t|+|\gamma_t| = n-1$. Furthermore, $|\beta_s| = s - 1$ and $|\beta_t| = t - 1$. As $s < t$, it must be the case that $|\beta_s| < |\beta_t|$. By our assumption $\widetilde{\alpha_{p_s}} = \widetilde{\alpha_{p_t}}$ and $\widetilde{\alpha_{q_s}} = \widetilde{\alpha_{q_t}}$. It follows that β_t has the form:

$$\beta_t = \beta_s, 0, \ldots \text{ because of (2) and (4)} \tag{6}$$

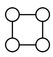
and

$$\beta_t = \beta_s, 1, \ldots \text{ because of (3) and (5)} \tag{7}$$

Because of the contradiction between (6) and (7), our assumption (1) is wrong.

We proved that for each value of k, such that $1 \leq k \leq n$, there is a distinct edge e_k in B^n and in H. It follows H has at least n edges. □

Problem 2. *Prove the n-dimensional hypercube graph is Hamiltonian for any $n \geq 2$.*

Solution:
By induction on n.
Basis: $n = 2$. Clearly, the graph  is Hamiltonian.
Induction hypothesis: For some $n \geq 2$, B^n is Hamiltonian.
Induction step: Consider B^{n+1}. Let $m = 2^n$. Let the vertex set of B^{n+1} be partitioned into V^0, the vectors having 0 in the leftmost position, and V^1, the vectors having 1 in the leftmost position. Let H^i be the subgraph of B^{n+1} induced by V_i, for $i = 0, 1$. It is known that both H^0 and H^1 are isomorphic to B^n. By the inductive hypothesis there is a Hamiltonian cycle c_0 in H_0. Clearly, $|c_0| = m$. Let

$$c_0 = u_1, u_2, \ldots, u_m$$

where $u_1, u_2, \ldots, u_m$ is some permutation of the vectors of V^0. Let v_i be the vector obtained from u_i by replacing the leftmost 0 by 1, for $1 \leq i \leq m$. Then $\{v_1, v_2, \ldots, v_m\} = V^1$. Furthermore,

$$c_1 = v_1, v_2, \ldots, v_m$$

is a Hamilton cycle in H^1.

Consider any edge $e = (u_k, u_{k+1})$ in c_0. Define that p_0 is the Hamilton path in H^0 obtained by removing e from c_0. Clearly, $e' = (v_k, v_{k+1})$ is an edge in c_1. Define that p_1 is the Hamilton path in H^1 obtained by removing e' from c_1. Note that (u_k, v_k) is an edge, call it e_k, in B^{n+1} because by construction the vectors u_k and v_k differ only at the leftmost position. Likewise, (u_{k+1}, v_{k+1}) is an edge, call it e_{k+1}, in B^{n+1}.

The paths p_0 and p_1 together with the edges e_k and e_{k+1} form a Hamilton cycle in B^{n+1}. □